A Proof of Poincaré's lemma

Ruihan He

November 2022

1 Introduction

1.1 Poincaré's lemma

Poincare's lemma:
If $U \subset \mathbb{R}^n$ is a star-shaped open set,H^p(U)=0, for ∀p>0

1.2 Pushback

1.2.1 Design

Suppose a map $f : V \to W$in manifold N, $f^* : Alt^*(W) \to Alt^*(V)$,if$(f^*\omega)(\xi_1, \cdots, \xi_p) := \omega(f(\xi_1), \cdots, f(\xi_p))$,the f$^* is Pushback. It looks like an inverse function with it's primitive function.$

1.2.2 Lemma

Suppose $f : V \to W$ is a linear map,$\omega^p(W), \eta^(W)$,
then$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

1.2.3 Proof

Let $\forall(\xi_1, \cdots, \xi_{p+q}) \in V^{p+q}$
$f^*(\omega \wedge \eta)(\xi_1, \cdots, \xi_{p+q}) = (\omega \wedge \eta)(f(\xi_1), \cdots, f(\xi_{p+q}))$
$= \sum_{\sigma \in S(p,q)} sign(\sigma)\omega(f(\xi_\sigma(1), \cdots, f(\xi_{\sigma(p)})))\eta(f(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)}))$
$= \sum_{\sigma \in S(p,q)} sign(\sigma) f^*\omega(\xi_{\sigma(1)}, dots, \xi_{\sigma(p)}) f^*\eta(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)})$
$= (f^*\omega^*\eta)(\xi_1, \cdots, \xi_{p+q})$
$\Rightarrow f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

1.2.4 Pushback in differential

ϕ's differential $D_x\phi : \mathbb{R}^n \to \mathbb{R}^m$ (linear map),Define the pullback $(D_x\phi)^* : Alt^p(\mathbb{R}^m) \to Alt^p(\mathbb{R}^n)$ for $\phi : U_1 \to U_2, \omega \in \Omega^p(U_2)$,and $\phi^*(\omega) : U_1 \to Alt^p(\mathbb{R}^n)$ is $(\phi^*(\omega))_x := (D_x\phi)^*(\omega_{\phi(x)})$
$\forall \xi_1, \cdots, \xi_p \in \mathbb{R}^n$,$(\phi^*(\omega))_x(\xi_1, \cdots, \xi_p) := \omega_{\phi(x)}((D_x\phi)(\xi_1), \cdots, (D_x\phi)(\xi_p))$
Example1 1-form:Consider Coordinate functions $y_i(y_i(\sum c_j \widetilde{e_j}) = c_i)$ in $\mathbb{R}^m$
$\omega = dy_i \in \Omega^1(U_2)$

$(\phi^*(dy_i))_x(\xi) = (dy_i)_{\phi(x)}((D_x\phi)(\xi))$
=dy$_i(\sum_k d\phi_k(\xi)\widetilde{e_k}) = \sum_k d\phi_k(\xi)\delta_{ik} \overset{i=k}{=} d\phi_i(\xi) \Rightarrow \phi^*(dy_i) = d\phi_i$
Example2 0-form:$f \in \Omega^0(U_2), Let \phi^* f := f \circ \phi$. If $\phi = (\phi_1, \cdots, \phi_m)$,
then $\phi^*(dy_j) = d\phi_j = \sum_i \frac{\partial \phi_j}{\partial x_i} dx_i$

2 Proof

Suppoese U is a star-shaped about the origin $0 \in \mathbb{R}^n$

The equivalent propositions of Poincaré is existing a linear operator$S_p : \Omega^p(U) \to \Omega^{p-1}(U)$ such that $d \circ S_p + S_{p+1} \circ d = id$ for $p > 0$

Suppose $\omega \in \Omega^p(U)$ and $d\omega = 0, p > 0$,by the equivalent propositions,$\omega = d(S_p(\omega)) + S_{p+1}(d\omega) = d(S_p(\omega)) \in d\Omega^{p-1}(U)$

Since $d(S_p(\omega)) = \omega$ for a closed p-form. All closed p-form are exact form,and $H^p(U) = 0$

Define $\overset{\wedge}{S_p} : \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U)$

For $\forall \omega \in \Omega^p(U \times \mathbb{R})$,having $\omega = \sum_I f_I(x,t)dx_I + \sum_J g_J(x,t)dt \wedge dx_J$ where I=(i$_1, \cdots, i_p), 1 \leqslant i_1 < \cdots < i_p \leqslant n, J = (j_1, \cdots, j_{p-1}), 1 \leqslant j_1 < \cdots < j_{p-1} \leqslant n$,and$f_I, g_J \in C^\infty(U \times \mathbb{R})$.Especially,the first one doesn't contain t(dx_I),and the second is the opposite($dt \wedge dx_J$).

Let $\overset{\wedge}{S_p}(\omega) := \sum_J (\int_0^{1) g_J(x,t)dt)dx_J}$

then $d\overset{\wedge}{S_p}(\omega) + \overset{\wedge}{S_{p+1}}(d\omega) = \sum_J d(\int_0^1 g_J(x,t)dt) \wedge dx_J + \overset{\wedge}{S_{p+1}}(\sum_I \frac{\partial f_I}{\partial t} dt \wedge dx_I + \sum_{J,i} \frac{\partial g_J}{\partial x_i} dx_i \wedge dt \wedge dx_J + dxterms)$

$=\sum_{J,i} \int_0^1 (\frac{J}{\partial x_i}(x,t)dt)dx_i \wedge dx_J + \sum_I (\int_0^1 \frac{\partial f_I}{\partial t} dt)dx_I - \sum_{J,i} \int_0^1 (\frac{J}{\partial x_i}(x,t)dt)dx_i \wedge dx_J$

$=\sum_I f_I(x,1)dx_I - \sum_I f_I(x,0)dx_I$

Define a map $\phi : U \times \mathbb{R} \to U, \phi(x,t) := \psi(t)x$

and $\psi(x) \in C^(\infty)(\mathbb{R}), 0 \leqslant \psi(t) \leqslant 1, \psi'(t) \geq 0$,

and $\psi(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ 1 & \text{if} \quad t \geq 1 \\ \psi(x) & \text{if} \quad 0 < t < 1 \end{cases}$

For $\omega = \sum_I h_I(x)dx_I \in \Omega^p(U)$

$\phi^*\omega = \sum_I h_I, (\psi(t)x)d(\psi(t)x_{i_1}) \wedge \cdots \wedge d(\psi(t)x_{i_p})$

$=\sum_I h_I(\phi(t)x)(\psi'(t)x_{i_1}dt + \psi(t)dx_{i_1}) \wedge \cdots \wedge (\psi'(t)x_{i_p}dt + \psi(t)dx_{i_p})$

$=\psi^p(t) \sum_I h_I(\psi(t)x)dx_I + \sum_J g_J(x,t)dt \wedge dx_J$

By the result of S_p,

d$S_p(\phi^*\omega) + S_{p+1}(d\phi^*\omega)$

$=\psi^p(1) \sum_I h_I(\psi(1)x)dx_I - \psi^p(0) \sum_I h_I((0)x)dx_I$

$=\sum_I h_I(x)dx_I = \omega$

Define $S_p := S_p \circ \phi^*$and$S_p : \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U)$.Noticed $d\phi^* = \phi^* d$

then $dS_p(\omega) + S_{p+1}(d\omega) = doversetS_p(\phi^*\omega) + oversetS_{p+1}(d\phi^*\omega) = \omega$