Homework

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问题 1

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求证：$w \in Alt^k(V) \Leftrightarrow w(v_1, \cdots, v_k) = 0$, 当 $v_i = v_{i+1}, i = 1, \cdots, k-1$ 时

Proof

Proof: "$\Longrightarrow$" 若 $w \in Alt^k(V)$, 则显然有： $w(v_1, \cdots, v_k) = 0$, 当 $v_i = v_{i+1}, i = 1, \cdots, k-1$ 时.
"$\Longleftarrow$" 注意到 $S(k)$ 由对换 $(i, i+1)$ 生成，且：

$$w(v_1, \cdots, v_i, v_{i+1}, \cdots, v_k) = w(v_1, \cdots, v_{i+1}, v_i, \cdots, v_k) \tag{1}$$

由此，对于任意的 $v_i = v_j, i \neq j, \exists \tau \in S(k), s.t.$

$$\tau(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = (v_1, \cdots, v_i, v_{i+1}, \cdots, v_k) \tag{2}$$

这里， $v_{i+1} = v_j = v_i$

$$\therefore w(v_1, \cdots, v_k) = sign(\tau) \cdot w_1(v_1, \cdots, v_i, v_{i+1}, \cdots, v_k) = 0 \tag{3}$$

$$\therefore w \in Alt^k(V) \tag{4}$$

□

问题 2-Christoffel 记号的推导 (Question17.7)

求证：$\Gamma_{ij}^{k}=\frac{1}{2}g^{kl}(\frac{\partial g_{il}}{\partial x^{j}}+\frac{\partial g_{lj}}{\partial x^{i}}-\frac{\partial g_{ij}}{\partial x^{l}})$

Proof

我们首先有：

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{il} \tag{5}$$

我们对指标 i,j,k 进行轮换，则有：

$$\frac{\partial g_{jk}}{\partial x^i} = \Gamma^l_{ji} g_{lk} + \Gamma^l_{ki} g_{jl} \tag{6}$$

$$\frac{\partial g_{ki}}{\partial x^j} = \Gamma^l_{kj} g_{li} + \Gamma^l_{ij} g_{kl} \tag{7}$$

Proof

注意到 $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}, g_{ij} = g_{ji}$,(5)+(6)-(7) 有:

$$\Gamma_{ik}^{l} g_{jl} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^{k}} + \frac{\partial g_{jk}}{\partial x^{i}} - \frac{\partial g_{ki}}{\partial x^{j}}\right) \tag{8}$$

在左右两边同乘 g^{pj}, 则有:

$$\Gamma_{ki}^{l} = \frac{1}{2} g^{jl}\left(\frac{\partial g_{ij}}{\partial x^{k}} + \frac{\partial g_{jk}}{\partial x^{i}} - \frac{\partial g_{ki}}{\partial x^{j}}\right) \tag{9}$$

□

问题 3-Poisson 括号积的局部表达式 Question17.1)

设

$$X=\sum_{i=1}^{m}a_i\frac{\partial}{\partial x_i},\ Y=\sum_{j=1}^{m}b_j\frac{\partial}{\partial x_j} \tag{10}$$

求证：

$$[X,Y]=\sum_{k=1}^{m}(\sum_{i=1}^{m}a_i\frac{\partial b_k}{\partial x_i}-\sum_{j=1}^{m}b_j\frac{\partial a_k}{\partial x_j})\frac{\partial}{\partial x_k} \tag{11}$$

我们有：

$$
\begin{aligned}
XY(f) &= \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{m} b_k \frac{\partial f}{\partial x_k}\right) \\
&= \sum_{k=1}^{m}\sum_{i=1}^{m}\left(a_i \frac{\partial b_k}{\partial x_i}\frac{\partial f}{\partial x_k} + a_i b_k \frac{\partial^2 f}{\partial x_k \partial x_i}\right) \\
&= \sum_{k=1}^{m}\left(\sum_{i=1}^{m} a_i \frac{\partial b_k}{\partial x_i}\right)\frac{\partial f}{\partial x_k} + \sum_{k=1}^{m}\sum_{i=1}^{m}\left(a_i b_k \frac{\partial^2 f}{\partial x_k \partial x_i}\right)
\end{aligned}
\tag{12}
$$

Proof

由于 X 和 Y 的地位相等，因而我们有：

$$\begin{aligned}[X, Y]f &= XY(f) - YX(f) \\ &= \sum_{k=1}^{m}\left(\sum_{i=1}^{m} a_i \frac{\partial b_k}{\partial x_i}\right)\frac{\partial f}{\partial x_k} + \sum_{k=1}^{m}\sum_{i=1}^{m}\left(a_i b_k \frac{\partial^2 f}{\partial x_k \partial x_i}\right) \\ &- \sum_{k=1}^{m}(\sum_{j=1}^{m} b_j \frac{\partial a_k}{\partial x_j})\frac{\partial f}{\partial x_k} - \sum_{k=1}^{m}\sum_{j=1}^{m}(b_j a_k \frac{\partial^2 f}{\partial x_k \partial x_j}) \\ &= \sum_{k=1}^{m}(\sum_{i=1}^{m} a_i \frac{\partial b_k}{\partial x_i} - \sum_{j=1}^{m} b_j \frac{\partial a_k}{\partial x_j})\frac{\partial f}{\partial x_k}\end{aligned} \tag{13}$$

$$\therefore [X, Y] = \sum_{k=1}^{m}(\sum_{i=1}^{m} a_i \frac{\partial b_k}{\partial x_i} - \sum_{j=1}^{m} b_j \frac{\partial a_k}{\partial x_j})\frac{\partial}{\partial x_k} \tag{14}$$

□

The End,Thank You!