problem 4.1,4.2 from SHU HAO ZHE

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1 Chain complexes and their Homology

1.1 Consider a commutative diagram of vectors spaces and linear maps with exact rows. Suppose that f_4 is injective, f_1 is surjective and f_2 is injective. Show that f_3 is injective. Suppose that f_2 is surjective, f_4 is surjective and f_5 is injective. Show that f_3 is surjective. In particular, we have that if f_1, f_2, f_4 and f_5 are isomorphisms, then f_3 is an isomorphism. (This assertion is called the 5-lemma.)

Proof. (idea: diagram chasing) We only need to prove the first statement since the other two can be solved using similar method. Set $x \in A_3$ such that $f_3(x) = 0$. the only thing we need to do is to examine whether $x = 0$ holds. Using property of commutative diagram, we obtain $b_3(f_3(x)) = 0 = f_4(a_3(x))$. f_4 is injective, so $a_3(x) = 0$, or $x \in Kera_3 = Ima_2$. There exists a element $x' \in A_2$ such that $a_2(x') = x$ and $f_3(a_2(x')) = b_2(f_2(x')) = 0$ which means $f_2(x') \in Kerb_2 = Imb_1$. So there's also $x'' \in B_1$ satisfying that $b_1(x'') = f_2(x')$. While f_1 is surjective, we can find an element $x''' \in A_1$ such that $f_1(x''') = x''$ and $b_1(f_1(x''')) = f_2(a_1(x'''))$ holds for sure. Because f_2 is an injection, so we can obtain $x' = a_1(x''')$ and finally, $x = a_2(x') = a_2(a_1(x''')) = 0$.

1.2 Consider the following commutative diagram where the rows are exact sequences. Show that there exists a exact sequence

$$0 \to Kerf_1 \to Kerf_2 \to Kerf_3 \to Cokf_1 \to Cokf_2 \to Cokf_3 \to 0.$$

Proof. Construction of the long exact sequence is actually called the Snake Lemma and proof of this profound lemma is done by three steps. First, show that $0 \to Kerf_1 \to Kerf_2 \to Kerf_3$ is exact. Second, prove that $Cokf_1 \to Cokf_2 \to Cokf_3 \to 0$ is also exact and this procedure is similar to the first step which is relatively easy. The last as well as hardest step is to construct a homomorphism from $Kerf_3$ to $Cokf_1$. You can see precise proof from any textbook concerning about category theory.

2 Homotopy

2.1 Show that "homotopy equivalence" is an equivalence relation in topological spaces.

Proof. Just need to prove that "homotopy equivalence" is reflexice, symmetric and transitive. (1) It's trivial that homotopy equivalence relation is reflexice (We can let the continuous map $F(x,t) = f(x)$.) (2) If $F(x,t)$ is a homotopy from f_0 to f_1, we can simply let another continuous map $G(x,t) = F(x, 1-t)$ which is a homotopy from f_1 to f_1 inversely. (3) Let $F(x,t)$ is a homotopy from f_0 to f_1, and $G(x,t)$ is another homotopy from f_1 to f_2. To prove the relation is transitive, we need to construct a homotopy from f_0 to f_2 which can be written as

$$H(x,t) = \begin{cases} F(x,2t), 0 \le t \le \dfrac{1}{2} \\ G(x,2t-1), \dfrac{1}{2} \le t \le 1 \end{cases} \tag{1}$$

We can obtain that "homotopy equivalence" is an equivalence relation.

2.2 Show that all continuous maps $f : U \to V$ that are homotopic to a constant map induce the 0-map $f^* : H^p(V) \to H^p(U)$ for $p > 0$.

Proof. Since $f : U \to V$ that are homotopic to a constant map which has a equivalent statement saying that the map is homotopic to identity map id. Using the same construction in the proof of the Poincaré Lemma,

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3 Exercise by ZHĀNG BóYuǎn

7.3. Show that there is no continuous map g :D^n→S^{n-1} with $g|_{S^{n-1}} \simeq id_{S^{n-1}}$.

Proof.

Assume that n≥ 2.

For the map r :R$^n\backslash\{0\} \to R^n\backslash\{0\}$,$r(x) = \frac{x}{\|x\|}$,we get that $id_{R^n\backslash 0}$,because $R^n\backslash 0$ always cotains the line segment vetween x and $r(x)$.

If g is of the indicated type,then $g(t \cdot r(x))$,$0 \le t \le 1$ defines a homotopy from a constant map to r.

This shows that $R^n\backslash\{0\}$ is contractible.

Since H^{n-1}(R$^n\backslash\{0\}$).

This contradicts *Poincar Lemma*.

8.2. Let $\varphi : N \to M$ be a continuous map from a smooth manifold N to a smooth submanifold M of R^k. Let i:$M \to R^k$ be the inclusion. Show that φ is smooth if and only if $i \circ \varphi$ is smooth.

Proof.

Assume that $\varphi = (\varphi_1, \cdots, \varphi_m), \varphi_s \in C^\infty$(N)$, s = 1, \cdots, m.$ i.e. φ is smooth.

And since $i = (x_1, \cdots, x_m, 0, \cdots, 0)$.

Then $i \circ \varphi = (\varphi_1, \cdots, \varphi_m, 0, \cdots, 0)$ is smooth.

Assume that $i \circ \varphi = (f_1, \cdots, f_k), f_s \in C^\infty$(M)$, s = 1, \cdots, k.$ i.e. $i \circ \varphi$ is smooth.

And we have map $p : R^k \to M, (x_1, \cdots, x_k) \mapsto (x_1, \cdots, x_m)$,

notice that $f_1 = \varphi_1, \cdots, f_m = \varphi_m$,

therefore $\varphi = p \circ i \circ \varphi$ is smooth.

8.6. Let $p_0 \in S^n$ be the "north pole" $p_0 = (0, \cdots, 0, 1)$.Show that $S^n\backslash\{p_0\}$ is deffeomorphic to R^n under stereographic projection, i.e. the map $S^n\backslash\{0\} \to R^n$ that carries $p \in S^n$ into the point of intersection between the line through p_0 and p and the equatorial hyperplane $R^n \subseteq R^{n+1}$.

Proof.

∀p∈ S^n,assume $p = (x_1, \cdots, x_n, x_{n+1})$.

And assume $q = (y_1, \cdots, y_n)$ is the corresponding point of p.

Then we have $\frac{y_1}{x_1} = \cdots = \frac{y_n}{x_n} = \frac{1}{1-x_{n+1}}$,

therefore $y_i = \frac{x_i}{1-x_{n-1}}$,$i = 1, \cdots, n$.

Because $p \in S^n$,$x_1^2 + \cdots + x_{n+1}^2 = 1$,

therefore $y_1^2 + \cdots + y_n^2 = \frac{x_1^2+\cdots+x_n^2}{(1-x_{n+1})^2} = \frac{1-x_{n+1}^2}{(1-x_{n+1})^2} = \frac{1+x_{n+1}}{1-x_{n+1}}$,

then compute and we know that $x_{n+1} = \frac{\sum_{i=1}^n y_i^2 - 1}{\sum_{i=1}^n y_i^2 + 1}$,

therefore $x_i = \frac{2y_i}{\sum_{i=1}^n y_i^2 - 1}$,which means that $S^n\backslash\{p_0\}$ is deffeomorphic to R^n.

4 Exercise by xu yi

2.7. Show for $g : V \to W$ that

$$Alt^{p+q}(f)(\omega_1 \wedge \omega_2) = Alt^p(f)(\omega_1) \wedge Alt^q(f)(\omega_2)$$

, where $\omega_1 \in Alt^p(W), \omega_2 \in Alt^q(W)$.

Proof.$Alt^{p+q}(f)(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \cdots, \xi_{p+q})$
$= (\omega_1 \wedge \omega_2)(f(\xi_1), f(\xi_2), \cdots, f(\xi_{p+q}))$
$= \sum_{\sigma \in S_(p,q)} sign(\sigma)\omega_1(f(\xi_{\sigma(1)}), f(\xi_{\sigma(2)}), \cdots, f(\xi_{\sigma(p)}))\omega_2(f(\xi_{\sigma(p+1)}), f(\xi_{\sigma(p+2)}), \cdots, f(\xi_{\sigma(p+q)}))$
$= \sum_{\sigma \in S_(p,q)} sign(\sigma)Alt^p(f)\omega_1(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)})Alt^q(f)\omega_2(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)})$
$= (Alt^p(f)(\omega_1) \wedge Alt^q(f)(\xi_1, \xi_2, \cdots, \xi_{p+q})$
therefore $Alt^{p+q}(f)(\omega_1 \wedge \omega_2) = Alt^p(f)(\omega_1) \wedge Alt^q(f)(\omega_2)$.

2.10. Let V be a 4-dimensional vector space and $\epsilon_1, \cdots, \epsilon_4$ a basis of $Alt^1(V)$.Let $A = (a_{ij})$ be a skew-symmetric matrix and define

$$\alpha = \sum_{i<j} a_i j \epsilon_i \wedge \epsilon_j.$$

Show that $\alpha \wedge \alpha = 0 \Leftrightarrow det(A) = 0$. Say $\alpha \wedge \alpha = \lambda \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4$.What is the relation between λanddet(A)?

Proof.$A = (a_{ij})$ be a skew-symmetric matrix
$a_{ii} = 0, a_{ij} = -a_{ji}, i \neq j$
$\alpha \wedge \alpha = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} + a_{23}a_{14} - a_{24}a_{13} + a_{34}a_{12})\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4$
$= 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4$
$det(A) = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2$
$\alpha \wedge \alpha = 0 \Leftrightarrow a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0 \Leftrightarrow det(A) = 0$.
and if $\alpha \wedge \alpha = \lambda \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4$, then $det(A) = (\frac{\lambda}{2})^2$.

5.3. Can R^2 be written as $R^2 = U \cup V$ where U, V are open connected sets such that $U \cap V$ is disconnected?

Proof. U, V are open connected sets ,hence

$$H^p(U) = \begin{cases} R & p = 0 \\ 0 & p > 0 \end{cases} \tag{2}$$

$$H^p(V) = \begin{cases} R & p = 0 \\ 0 & p > 0 \end{cases} \tag{3}$$

$$H^p(R^2) = \begin{cases} R & p = 0 \\ 0 & p > 0 \end{cases} \tag{4}$$

by the poincare lemma.
From the Mayer-Vietoris sequence we have

$$0 \to H^0(R^2) I^*_{\to} H^0(U) \oplus H^0(V) J^*_{\to} H^0(U \cap V) \delta^*_{\to} H^1(R^2) \to \cdots$$

.

Therefore $H^0(U \cap V) = ImJ^* = H^0(U) \oplus H^0(V)/KerJ^* = R \oplus RImI^* = R \oplus R/R = R$,$U \cap V$ is connected .