From Calculus to Cohomology

Homework 3

Exercise 1. Prove the Cartan lemma.

Exercise 2. Let V be an n-dimensional vector space with inner product $\langle , \rangle$. A volume element of V is a unit vector $vol \in Alt^n(V)$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V with $vol(e_1, \ldots, e_n) = 1$, and $\{\epsilon_1, \ldots, \epsilon_n\}$ the dual basis of $Alt^1(V)$. Define the Hodge star operator $*\colon Alt^p(V) \to Alt^{n-p}(V)$ as a linear map determined by

$$*(\epsilon_{\sigma(1)} \wedge \ldots \wedge \epsilon_{\sigma(p)}) = sgn(\sigma)\epsilon_{\sigma(p+1)} \wedge \ldots \wedge \epsilon_{\sigma(n)},$$

for any $\sigma \in S(p, n-p)$. Show that the composition $* \circ *\colon Alt^p(V) \to Alt^p(V)$ is simply a multiplication by $(-1)^{p(n-p)}$.