

Relative quantum cohomology under birational transformations

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@ SWJTU

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 - Absolute Gromov–Witten theory
 - The crepant/discrepant transformation conjecture
- 2 Relative Gromov–Witten theory
 - Definition
 - Relative Quantum Cohomology
 - Simple normal crossings pairs
- 3 Relative Gromov–Witten theory under birational transformations
 - Set-up
 - Toric wall-crossings
- 4 Connection to extremal transitions
- 5 Connection to FJRW theory

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Gromov–Witten theory

Gromov–Witten theory is a curve counting theory. However, instead of studying curves, we study maps from curves to the target variety.

Moduli Space of Stable Maps

Let X be a smooth projective variety.

The moduli space $\overline{M}_{g,n}(X, d)$ of stable maps of degree d from genus g nodal curves with n -markings to X consists of

$$(C, \{p_i\}_{i=1}^n) \xrightarrow{f} X,$$

where

- C is a projective, connected, nodal curve of genus g ;
- p_1, \dots, p_n are distinct nonsingular points of C ;
- $f_*[C] = d \in H_2(X)$;
- stable: automorphisms of the map is finite

Definition

Given cohomological classes $\gamma_i \in H^*(X)$, one can define the Gromov–Witten invariant

$$\left\langle \prod_{i=1}^n \tau(\gamma_i) \right\rangle_{g,n,d}^X := \int_{[\overline{M}_{g,n}(X,d)]^{\text{vir}}} \prod_{i=1}^n (\text{ev}_i^* \gamma_i),$$

Quantum Cohomology

The quantum cohomology ring $QH^*(X)$ is a deformation of the usual cohomology ring using Gromov–Witten invariants.

Quantum Product

Given $\alpha, \beta \in H^*(X)$, the quantum product is defined using three-point Gromov–Witten invariants.

$$\alpha \circ \beta = \sum_{d \in H_2^{\text{eff}}(X)} \sum_k Q^d \langle \alpha, \beta, \phi_k \rangle_{0,3,d}^X \phi^k$$

where $\{\phi_k\}$ and $\{\phi^k\}$ are dual basis of $H^*(X)$.

Enumerative mirror symmetry

- The A-model data is a generating function of genus zero Gromov–Witten invariants called the J -function $J_X(\tau, z)$.
- The B-model data is period integrals called the I -function $I_X(y, z)$

Mirror theorem (Givental 1996, Lian–Liu–Yau 1997, ...)

$$J_X(\tau(y), z) = I_X(y, z)$$

where $\tau(y)$ is called the mirror map.

For quintic threefold the I -function is

$$\begin{aligned} I_X(y) &= \sum_{d \geq 0} y^{H+d} \frac{\prod_{a=1}^{5d} (5H+a)}{\prod_{a=1}^d (H+a)^5} \\ &= \sum_{d \geq 0} \frac{(5d)!}{(d!)^5} y^d H^0 + O(H). \end{aligned}$$

Remark

Quantum cohomology can be reconstructed from the I -function.

- Gromov–Witten invariants are deformation invariants, but not birational invariants.

Natural Question

How Gromov–Witten theory varies under birational transformations?

The crepant transformation conjecture (by Yongbin Ruan)

Given a birational transformation $\phi: X_+ \dashrightarrow X_-$. Suppose there is a \tilde{X} with projective birational morphisms $f_{\pm}: \tilde{X} \rightarrow X_{\pm}$ such that the following diagram commute

$$\begin{array}{ccc} & \tilde{X} & \\ f_+ \swarrow & & \searrow f_- \\ X_+ & \overset{\phi}{\dashrightarrow} & X_- \end{array}$$

and

$$f_+^* K_{X_+} = f_-^* K_{X_-}.$$

Then the quantum cohomology

$$\text{QH}(X_+) = \text{QH}(X_-)$$

under analytic continuations.

The crepant transformation conjecture

- Ruan (2006)
- Coates–Ruan (2013)
- Lee–Lin–Wang (2010, 2016, 2019,...)
- Gonzalez–Woodward (2012)
- Coates–Iritani–Jiang (2018)
- ...

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Coates–Iritani–Jiang (2018)

For toric complete intersections, the I -functions of X_+ and X_- are related by analytic continuation.

The discrepant transformation

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If $f_+^* K_{X_+} \neq f_-^* K_{X_-}$, then ϕ is discrepant.

- Iritani (2020)
- Acosta–Shoemaker (2018, 2020)

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The Laplace transform of the regularized I -function of X_- can be analytically continued to the I -function of X_+ .

Question

The relation in the discrepant case is more complicated and more difficult to obtain. Is there an easier approach to this question?

Turning into log crepant

We consider divisors $D_+ \subset X_+$ and $D_- \subset X_-$ such that

$$f_+^*(K_{X_+} + D_+) = f_-^*(K_{X_-} + D_-).$$

Then compare the relative quantum cohomology of the pairs (X_+, D_+) and (X_-, D_-) .

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Relative Gromov–Witten theory is the enumerative theory of counting curves with tangency condition along a divisor (a codimension one subvariety).

- X : a smooth projective variety.
- D : a smooth divisor of X .
- For $d \in H_2(X, \mathbb{Q})$, we consider a partition $\vec{k} = (k_1, \dots, k_m)$ of $\int_d [D]$. That is,

$$\sum_{i=1}^m k_i = \int_d [D], \quad k_i > 0$$

- $\overline{M}_{g, \vec{k}, n, d}(X, D)$: the moduli space of $(m+n)$ -pointed, genus g , degree $d \in H_2(X, \mathbb{Q})$, relative stable maps to (X, D) such that the relative conditions are given by the partition \vec{k} .

Evaluation Maps

There are two types of evaluation maps.

$$\text{ev}_i : \overline{M}_{g, \vec{k}, n, d}(X, D) \rightarrow D, \quad \text{for } 1 \leq i \leq m;$$

$$\text{ev}_i : \overline{M}_{g, \vec{k}, n, d}(X, D) \rightarrow X, \quad \text{for } m+1 \leq i \leq m+n.$$

The first m markings are relative markings with contact order k_i , the last n markings are interior markings.

Data

- $\delta_i \in H^*(D, \mathbb{Q})$, for $1 \leq i \leq m$.
- $\gamma_{m+i} \in H^*(X, \mathbb{Q})$, for $1 \leq i \leq n$.

Definition

Relative Gromov–Witten invariants of (X, D) are defined as

$$\left\langle \prod_{i=1}^m \tau(\delta_i) \left| \prod_{i=1}^n \tau(\gamma_{m+i}) \right. \right\rangle_{g, \vec{k}, n, d}^{(X, D)} := \int_{[\overline{M}_{g, \vec{k}, n, d}(X, D)]^{\text{vir}}} \prod_{i=1}^m \text{ev}_i^*(\delta_i) \prod_{i=1}^n \text{ev}_{m+i}^*(\gamma_{m+i}). \quad (1)$$

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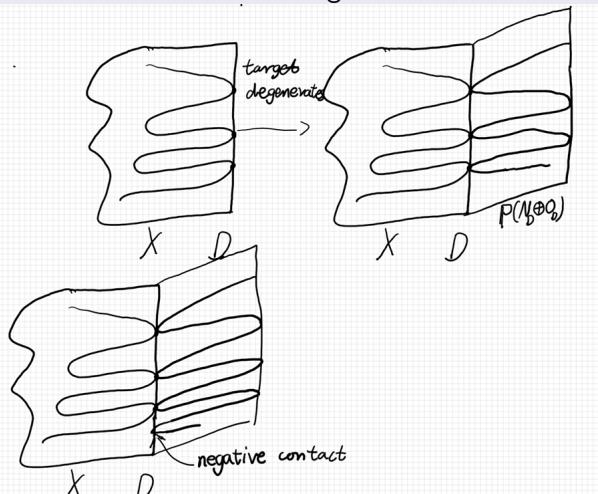
However

To have relative quantum cohomology, we need more than positive contact orders.

Negative contact invariants

Fan–Wu–Y, 2018 & 2019

Relative Gromov–Witten invariants with negative contact orders are defined to count the following curves.



Relative quantum cohomology

Relative Gromov–Witten invariants with (possibly) negative contact orders have the following property

- Relative quantum cohomology ring

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Remark

These relative invariants of (X, D) coincide with orbifold invariants of the root stack $X_{D,r}$ when $r \gg 1$.

$$\langle \dots \rangle^{(X,D)} = \langle \dots \rangle^{X_{D,r}}$$

[Abramovich–Cadman–Wise, 2017] and [Fan–Wu–Y, 2018]

Theorem (Fan–Tseng–Y, 2018)

The I -function for the pair (X, D) is

$$I_{(X,D)}(Q, t, z) = \sum_{d \in \overline{NE}(X)} J_{X,d}(t, z) Q^d \left(\prod_{0 < a \leq D \cdot d - 1} (D + az) \right) [\mathbf{1}]_{-D \cdot d}.$$

Invariants of simple normal crossing pairs

- X : smooth projective variety;
- $D = D_1 + \cdots + D_n$: simple normal crossings divisor;
- $\vec{r} = (r_1, \dots, r_n) \in (\mathbb{Z}_{>0})^n$;
- $X_{D, \vec{r}}$: multi-root stack of X along D .

Theorem (Tseng-Y, 2020)

For r_1, \dots, r_n sufficiently large,
genus 0: $\langle \rangle^{X_{D, \vec{r}}}$ is independent of r_1, \dots, r_n .

A Gromov–Witten invariants of snc pairs via orbifold

$X_{D,\infty}$: infinite root stack.

Definition (Tseng-Y, 2020)

The genus zero formal Gromov-Witten invariants of $X_{D,\infty}$ are defined as

$$\langle \dots \rangle^{X_{D,\infty}} := \langle \dots \rangle^{X_{D,\vec{r}}}$$

for sufficiently large \vec{r} .

Remark

There is a quantum cohomology ring for $X_{D,\infty}$. We also call relative quantum cohomology of the snc pair (X, D) .

Remark

The orbifold invariants of $X_{D,\infty}$ are different from log invariants in general. Log invariants are invariant under birational transformations, but orbifold invariants are not.

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The orbifold invariants of $X_{D,\infty}$ are different from log invariants in general. Log invariants are invariant under birational transformations, but orbifold invariants are not.

Question

Are there still some kinds of birational invariance (not on the level of single invariants) in orbifold Gromov–Witten theory of $X_{D,\infty}$?

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We consider divisors $D_+ \subset X_+$ and $D_- \subset X_-$ such that

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What is the relation between Gromov–Witten theories of the pairs (X_+, D_+) and (X_-, D_-) ?

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In the spirit of the crepant transformation conjecture, there should be some form of birational invariance for Gromov–Witten theory of (X, D) .

Example: root constructions

- X : smooth projective variety
- $D \subset X$: smooth divisor

Let $X_{D,r}$ be the r -th root stack of X along D . Then the natural map

$$X_{D,r} \rightarrow X$$

is a discrepant transformation.

Relation between absolute invariants

Tseng–Y, 2016: $GW(X_{D,r})$ is determined by $GW(X)$ and $GW(D)$ and the restriction map $H^*(X) \rightarrow H^*(D)$.

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Relation between relative invariants

Abramovich–Fantechi, 2016: $GW(X_{D,r}, D_r) = GW(X, D)$.

The relation between relative invariants are simpler than the relation between absolute invariants.

Example: blow-ups

Let X_+ be a blow-up of X_- along a complete intersection center $D_{-,1} \cap \cdots \cap D_{-,n}$.

- We can choose **snc divisors**

$$D_- = D_{-,1} + \cdots + D_{-,n}, \quad (2)$$

and

$$D_+ = D_{+,1} + \cdots + D_{+,n} + E, \quad (3)$$

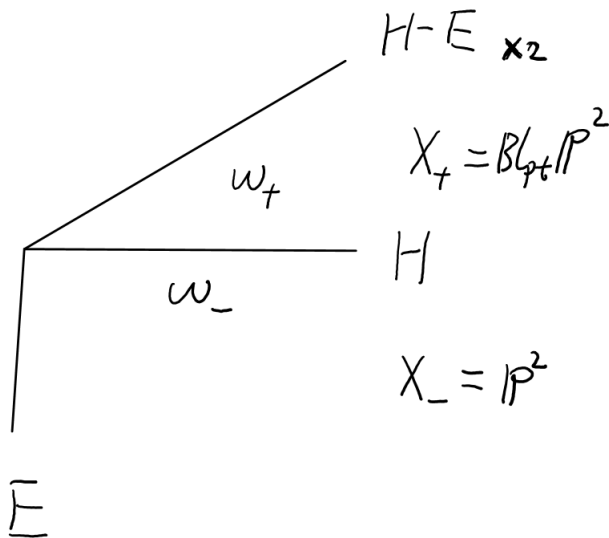
where $D_{+,i}$ are strict transform of $D_{-,i}$ and E is the exceptional divisor.

- We can also choose D_+ and D_- be **smooth divisors** that are linear equivalent to $D_{+,1} + \cdots + D_{+,n} + E$ and $D_{-,1} + \cdots + D_{-,n}$.

- Toric Deligne–Mumford stacks are defined as GIT quotients with a choice of stability condition.
- There is a wall and chamber structure in the secondary fan.
- A single wall-crossing gives a birational transformation between toric Deligne–Mumford stacks.
- For example, a toric blow-up along a complete intersection of toric divisors is given by a discrepant toric wall-crossing.

Toric wall-crossing

The secondary fan (GKZ fan)



Theorem (Y, 2022)

Given

- *a birational transformation $\phi : X_+ \rightarrow X_-$ between toric Deligne–Mumford stacks (or toric complete intersections) is given by a single toric wall-crossing.*
- *D_+ and D_- are simple normal-crossings divisors such that toric divisors containing the loci of indeterminacy are the irreducible components.*

Then their relative I-functions are directly identified (without analytic continuation).

Case 2) smooth divisors

- We need to assume the divisors are nef. Then we apply one of the following results.
- 1) Via local-relative correspondence.
- 2) Via relative-orbifold correspondence and the hyperplane construction of root stacks.

We will explain 1).

Theorem (van Garrel–Graber–Ruddat, 2019: smooth divisors)

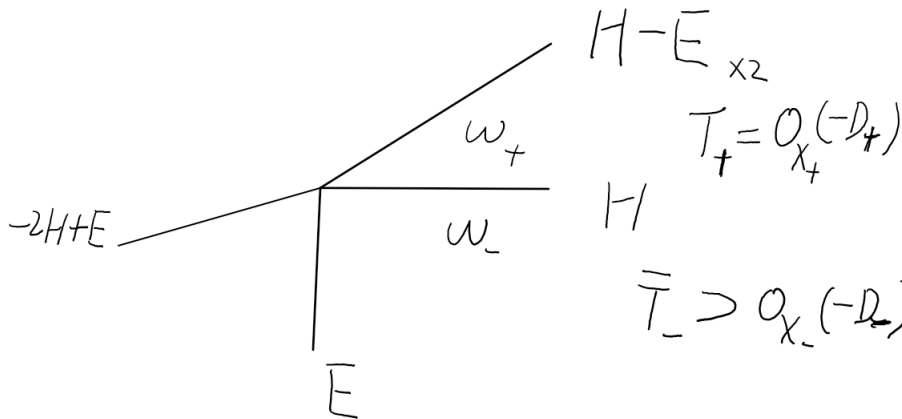
In genus zero with maximal contact, $GW(\mathcal{O}_X(-D))_0 = GW(X, D)_0$.

Theorem (Battistella–Nabijou–Tseng–Y, 2021: snc divisors)

In genus zero with maximal contact, $GW(\bigoplus_{i=1}^n \mathcal{O}_X(-D_i))_0 = GW(X_{D, \infty})_0$.

Relating local invariants

By the local-orbifold correspondence, we just need to compare local invariants of $\mathcal{O}_{X_+}(-D_+)$ and $\mathcal{O}_{X_-}(-D_-)$.



Relating local invariants

Then the result essentially follows from Mi-Shoemaker which states

Theorem (Mi-Shoemaker, 2020)

The narrow quantum D -modules of $\mathcal{O}_{X_+}(-D_+)$ and $\mathcal{O}_{X_-}(-D_-)$ are related by analytic continuation and specialization of a variable.

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Remark

Beyond maximal contacts, we need to use the relative-orbifold correspondence.

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Extremal transition

Two smooth projective varieties X and \tilde{X} are said to be related by extremal transition if they are related by a birational contraction and a smoothing.

Reid's fantasy

Any pair of smooth Calabi–Yau threefolds may be connected via a sequence of flops and extremal transitions.

Example

Cubic extremal transitions

- S : a cubic surface embedded in a smooth Calabi–Yau threefold \tilde{Y} such that the rational curves in S generate an extremal ray in the sense of Mori.
- Consider a birational contraction $\tilde{Y} \rightarrow X_0$ where S is contracted to a point with local equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

- Deform the local equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = t(t \neq 0)$$

to obtain another Calabi–Yau threefold Y .

Connection to extremal transitions

- Li–Ruan, 2001: Gromov–Witten invariants of conifold transition via the degeneration formula.
- Lee–Lin–Wang, 2018: $A+B$ theory in conifold transitions for Calabi–Yau threefolds.
- Rongxiao Mi, 2017: Gromov–Witten theory of cubic extremal transition via mirror symmetry.
- Mi–Shoemaker, 2020: Extremal transitions via quantum Serre duality.

The set-up of Mi-Shoemaker

Let $\tilde{X} \rightarrow X$ be a toric blow-up. By Mi-Shoemaker, there are hypersurfaces $\tilde{D} \subset \tilde{X}$ and $D \subset X$ such that \tilde{D} and D are related by extremal transitions. The pairs (\tilde{X}, \tilde{D}) and (X, D) are log-K-equivalent!

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Theorem (Mi-Shoemaker, 2020)

The ambient quantum D-modules of \tilde{D} and D are related by analytic continuation and specialization of a variable.

Proof via quantum Serre duality.

The ambient quantum D-modules of hypersurfaces D is equivalent to the narrow quantum D-module of $\mathcal{O}_X(-D)$. □

Example

Cubic extremal transitions

- Y : quintic threefold in \mathbb{P}^4
- $\tilde{X} := \text{Bl}_{\text{pt}} \mathbb{P}^4$.
- $\tilde{Y} \subset \tilde{X}$ is a hypersurface defined by the divisor $\tilde{D} = 5H - 3E$.

Rank reduction in extremal transitions

- There is a rank discrepancy between the ambient quantum D-modules of \tilde{Y} and Y : the rank of the ambient quantum D-module of \tilde{Y} is 6 and the rank of the ambient quantum D-module of Y is 4.
- There are two extra solutions of the Picard-Fuchs equation coming from the analytic continuation of the ambient quantum D-module of \tilde{Y} .

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- There are two extra solutions of the Picard-Fuchs equation coming from the analytic continuation of the ambient quantum D-module of \tilde{Y} .

Theorem (Mi, 2017)

For cubic extremal transitions, the rank reduction is partially explained as the FJRW theory of the cubic singularity: The two extra solutions, after specialization, recover the regularized FJRW theory of the cubic singularity.

Question

Why FJRW theory? Why not Gromov–Witten theory directly?

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Recall:

Theorem (Mi, 2017)

After analytic continuation

$$I_{\tilde{Y}}(y_1, y_2 = 0) = I_Y(y_1)$$

Rank reduction in extremal transitions

Question

Why FJRW theory? Why not Gromov–Witten theory directly?

Recall:

Theorem (Mi, 2017)

After analytic continuation

$$I_{\tilde{Y}}(y_1, y_2 = 0) = I_Y(y_1)$$

Theorem (Y, 2022)

The rank reduction is partially explained as the local Gromov–Witten theory of the total space K_S of the canonical bundle of the cubic surface S :

$$\iota^* I_{\tilde{Y}}(y_1 = 0, y_2) = I_{K_S}, \quad \text{where } \iota : S \hookrightarrow \tilde{Y}.$$

Rank reduction in extremal transitions

In general, suppose toric hypersurfaces $\tilde{D} \subset \tilde{X}$ and $D \subset X$ are related by extremal transitions. Let S be the subvariety of \tilde{D} that is contracted under the transition.

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Question

Why FJRW theory appears in [Mi, 2017]?

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Connection to FJRW theory

- Gromov–Witten theory of Calabi–Yau hypersurface X in a weighted projective space $\mathbb{P}[w_1, \dots, w_N]$.
- FJRW theory of a quasi-homogeneous polynomial W of degree $d = \sum_{i=1}^N w_i$ and a group $G = \langle J_W \rangle$, where

$$J_W := (\exp(2\pi i w_1/d), \dots, \exp(2\pi i w_N/d)) \in (\mathbb{C}^\times)^N.$$

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LG/CY correspondence

Gromov–Witten theory of $X = \{W = 0\}$ matches with FJRW theory of (W, G) via analytic continuation.

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- Chiodo–Ruan 2010.
- Chiodo–Iritani–Ruan 2014.
- Lee–Priddis–Shoemaker 2016.
- Clader–Ross 2018.
- Y. Zhao 2021.

Beyond Calabi–Yau condition

P. Acosta (2014): LG/(Fano, general type) correspondence.

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Recall

The rank reduction in cubic extremal transition can be partially explained as either the FJRW theory of cubic singularity or the local/relative Gromov–Witten theory of the cubic surface. How are they related?

Mirror symmetry for Fano variety

$$\text{Fano } X \quad \begin{array}{c} \xrightarrow{\text{Mirror}} \\ \xleftarrow{\quad} \end{array} \quad \text{Landau–Ginzburg model } (X^\vee, W)$$

For Fano/LG mirror symmetry, it is expected that the generic fiber of W : $W^{-1}(t)$ is mirror to the smooth anticanonical divisor of X . Therefore, it is more natural to consider mirror symmetry for a log Calabi–Yau pair (X, D) .

Log Calabi–Yau mirror symmetry

For a smooth log Calabi–Yau pair (X, D)

$$\text{Log Calabi–Yau } (X, D) \begin{array}{c} \xrightarrow{\text{Mirror}} \\ \xleftarrow{\hspace{1.5cm}} \end{array} \text{LG model } (X^\vee, W)$$

$$\text{Noncompact Calabi–Yau } X \setminus D \begin{array}{c} \xrightarrow{\text{Mirror}} \\ \xleftarrow{\hspace{1.5cm}} \end{array} X^\vee \text{ (without } W)$$

$$\text{Smooth anticanonical divisor } D \begin{array}{c} \xrightarrow{\text{Mirror}} \\ \xleftarrow{\hspace{1.5cm}} \end{array} W^{-1}(t) \text{ for generic } t \in \mathbb{C}$$

Given a Fano hypersurface $X \subset \mathbb{P}[w_1, \dots, w_N]$ with its smooth anticanonical divisor D . Is there a LG/log CY correspondence?

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Question

Is it a direct enumerative meaning of the regularized FJRW I -function?

Thank you!