# Remarks on 2d Unframed Quiver Gauge Theories

work with Hao Zou (BIMSA), to appear

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**ITP & BIMSA**

**Workshop on interaction between Geometric Topology and Mathematical Physics**
**Southwest Jiaotong University**
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# Plan

- Review of GLSM and Kähler quotient
  - Quiver varieties
  - Dualities as cluster mutations
- Positive GLSM quiver and infinite chain of dualities
  - Kronecker quiver
  - Markov quiver
- Abelian necklace quiver and 2d SQCD
  - Coulomb branch analysis
  - Quantum Coulomb branch

For the math audience, the goal is to draw attention to (unframed) quiver varieties as interesting objects to study Gromov-Witten theory.

For the physics audience, the goal is to present a connection between abelian quiver gauge theories and nonabelian theories in 2d.

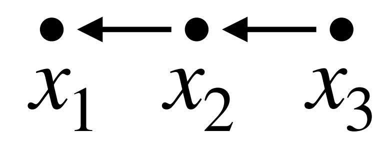
# Kähler Moduli and Cluster Algebra

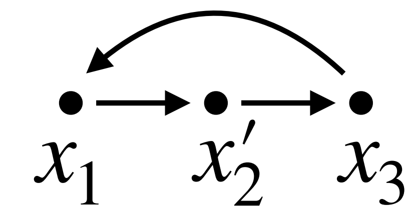
- 2d quiver GLSM provide an important class of Kähler quotient construction of quiver varieties

- In gauge theory, it is natural to think of a quiver gauge group with physical parameters as cluster variables.

- By studying dualities, We found a surprising cluster algebra structure on the Kähler moduli space

Benini-Park-PZ ‘14




GLSM A

- Gauge group $G$
- Matter field $\phi_i \in \mathbb{C}$
- moment map $\mu$

GLSM B

- Gauge group $G'$
- Matter field $\phi_i'$
- moment map $\mu'$

RG flow

Same IR observables (generating functions)

We apply Fan-Jarvis-Ruan’s theory to quiver varieties, before and after mutations, to obtain their generating functions $\mathscr{F}_g^{bf}, \mathscr{F}_g^{af}$. Recall that we have a quantum variable $q_i$ for each vertex. Benini-Park-Zhao’s physical analysis suggests the following mathematical conjecture.

**Mutation Conjecture** $\mathscr{F}_g^{bf}$ and $\mathscr{F}_g^{af}$ are equivalent up to the change of variables

$$\widetilde{q}_i = \begin{cases} q_k^{-1}, & \text{if } \ i = k, \\ q_i q^{|a_{ki}|}(q_k+1)^{-a_{ki}}, & \text{if } \ \text{Otherwise}, \end{cases}$$

Ruan ‘17

- In the first part of the talk, I will review the physical analysis that lead to a precise mathematical conjecture

# GLSM and Kähler quotient

- In any quantum field theory (or physical system, the central object is the space of ground states, or vacua.
- We are interested in gauged linear sigma models defined by a gauge group and chiral fields.
- Consider the U(1) theory with $n$ chirals, with potential

$$U = \sum_{i=1}^{n} |\sigma|^2 |\phi_i|^2 + \frac{e^2}{2}\left(\sum_{i=1}^{n} |\phi_i|^2 - r\right)^2$$

- If $r > 0$, then $\phi \neq 0$ and the **Higgs branch vacua** is geometrically $\mathbb{CP}^{n-1} = S^{2n-1}/U(1)$

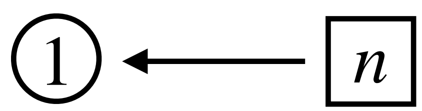
$$\mathbb{CP}^{n-1} = \left\{ (\phi_1, \ldots, \phi_n) \in \mathbb{C}^n \middle| \, |\sum_{i=1}^{n} |\phi_i|^2 = r \right\} / \mathrm{U(1)}$$

- If $r = 0$, then $\phi = 0$ and $\sigma \neq 0$, and we have the **Coulomb branch vacua** (more on this later)
- In the $e \to \infty$ limit, the GLSM is believed to flow to a nonlinear sigma model whose target space is the classical Higgs branch

Witten '93

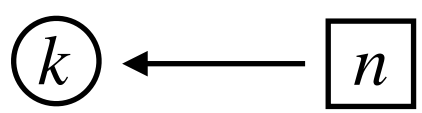
# GLSM and Kähler quotient

- This provides a large class of Kähler quotient manifolds such as Grassmannians, toric varieties and determinant Calabi-Yau's.
- One can also introduce superpotentials, that engineer hypersurfaces in toric varieties.
- It is useful to introduce a quiver notation. A quiver is a directed graph with nodes and arrows.
- A circle node represents a gauge group U(k), a square (frame) node represents a flavor group U(n) and an arrow represents a field $\phi_i^\alpha \in \mathrm{Mat}(\mathbb{C}^k, \mathbb{C}^n)$




Kähler parameter $r$

$$\sum_i^n |\phi_i|^2 = r \qquad \mathbb{CP}^{N-1} = S^{2N-1}/U(1)$$




$r$

$$\sum_i^n \phi_i^\alpha \phi_i^\beta = r\delta^{\alpha\beta} \qquad Gr(k, N)$$

Space of $k$-planes in $\mathbb{C}^N$

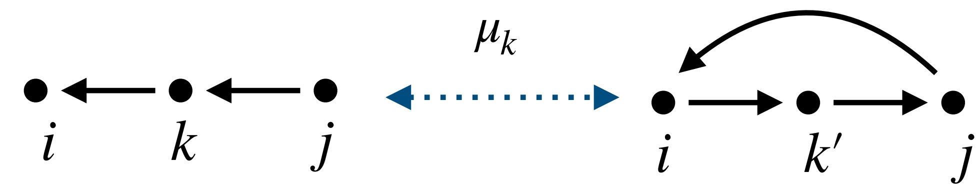
# Quiver varieties

- It is natural to consider quiver gauge theories with multiple Kähler parameters
- The mathematical framework is quiver varieties
  - Assign a gauge group GL($N_i$) to each gauge node
  - Each arrow defines a vector space $\mathbb{C}^{N_i \times N_j}$
  - $V = \bigoplus_{i \to j \in Q_1} \mathbb{C}^{N_i \times N_j}$ modulo the gauge group $G$ : $\prod_{\text{gauge nodes}}$ GL($N_i$)
  - The adjoint action on $G$ induces a momentum map $\mu : V \to \mathfrak{g}^*$
  - The quiver variety is the GIT quotient $\mu^{-1}(r_i)//G$

See e.g. Kirillov's textbook

Example: Flag variety

# Quiver mutations




- There is a local operation on a quiver known as mutation on the $k$ node
    - Reverse arrows emanating from $k$
    - For each path $i \to k \to j$ passing through $k$, add an arrow $j \to i$
    - Remove pairs of arrows forming a 2-cycle
- A GLSM quiver has additional data: the gauge node ranks $N_i$ and the complexified Kähler parameters $t_i = 2\pi r_i + i\theta_i$
- For a single gauge node, the Grassmannian duality is an example of quiver mutation

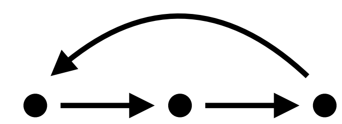

$$Gr(k, N) \quad \simeq \quad Gr(N-k, N)$$

# Seiberg-like dualities

- Quiver mutation were known to physicists as Seiberg duality

Seiberg '94

- $G' = \text{SU}(N)$
- $N_f$ "quarks" $\phi_i$
- $N_f$ "anti-quarks" $\tilde{\phi}_i$



- $G' = \text{SU}(N_f - N)$
- $N_f$ "quarks" $\Phi_i$
- $N_f$ "anti-quarks" $\Phi_i$
- $N_f N_a$ "mesons" $M_{i,j} = \phi_i \tilde{\phi}_j$
- Superpotential $\mathscr{W} = \text{Tr}(M_{i,j}\Phi_i\tilde{\Phi}_j)$

- 2d Seiberg-like dualities are similar, except we can have a different number of $N_f$ quarks and $N_a$ antiquarks, and $N' = \max(N_f, N_a) - N$

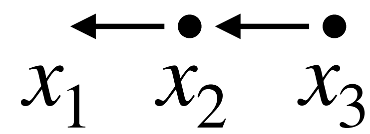
Hanany-Hori '97

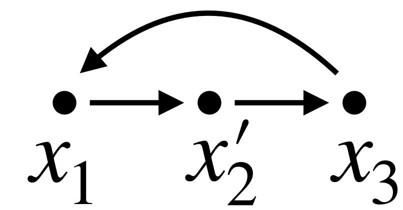
Hori-Tong '06

Benini-Cremonesi '12

# Duality as cluster transformations

- We studied how dualities act on a general quiver, and found the following transformation rules on the GLSM data




- The gauge group ranks transform as tropical cluster $x$-variables

$$N_i' = \max(N_i^{\text{in}}, N_i^{\text{out}}) - N_i$$

- We observe that the Kähler coordinates $z_i \sim e^{-t_i}$ mutate as dual cluster variables

$$z_i' = \begin{cases} z_k^{-1} & \text{if } i = k \\ z_i \, z_k^{[b_{ki}]_+}(1 + z_k)^{-b_{ki}} & \text{if } i \neq k \end{cases}$$

| GLSM A | GLSM B |
|---|---|
| • Gauge group $G$ | • Gauge group $G'$ |
| • Matter field $\phi_i \in \mathbb{C}$ | • Matter field $\phi_i'$ |
| • moment map $\mu$ | • moment map $\mu'$ |

RG flow

Same IR observables
(generating functions)

Benini-Park-PZ ‘14

# Partition function test of dualities

- Dualities can be tested by $\mathbb{S}^2$ partition functions, that factorizes onto sums of products of vortex and anti-vortex partition functions

$$Z^{N_f,N_a}_{U(N)} \sim \sum_{\vec{F}\in C(N,N_f)} Z_0^{\vec{F}}\, Z_{\mathrm{v}}^{\vec{F}}\, Z_{\mathrm{av}}^{\vec{F}}$$

$$Z_{\mathrm{v}}^{\vec{F}}\left(\Sigma_F, \tilde{\Sigma}_A; z\right) = \sum_{n\geq 0} z^n \sum_{(n_I)=n} \prod_{I=1}^{N} \frac{\prod_{A=1}^{N_a}\left(\Sigma_A^{F_I}\right)_{n_I}}{\prod_{J=1}^{N}\left(-\Sigma_{F_J}^{F_I} - n_I\right)_{n_J} \prod_{J'=1}^{N'}\left(-\Sigma_{F^c_{J'}}^{F_I} - n_I\right)_{n_I}}$$

- An identity between $Z_{\mathrm{v}}^{\vec{F}}$ defined on an $N$-tuple of vortex configurations and $Z_{\mathrm{v}}^{\vec{F}^c}$ on its complement leads to

$$Z^{N_f,N_a}_{U(N)}\left(\Sigma_{F\pm}, \tilde{\Sigma}_{A\pm}, r_F, \tilde{r}_A; z\right) = f^{(r)}_{\mathrm{imp}}\, f_{\mathrm{ctc}} \prod_{F,A} \frac{\Gamma\left(\Sigma^F_{A+} + \frac{r_F+\tilde{r}_A}{2}\right)}{\Gamma\left(1-\Sigma^F_{A-} - \frac{r_F+\tilde{r}_A}{2}\right)} \cdot Z^{N_a,N_f}_{U(N')}\left(\tilde{\Sigma}_{A\pm}, \Sigma_{F\pm}, 1-\tilde{r}_A, 1-r_F;\, z^{-1}\right)$$

- The contact term $f_{\mathrm{ctc}}$ affects the Kähler coordinates of neighboring gauge nodes and leads to the cluster transformation

# Mutation conjecture

- The generating functions of Gromov-Witten invariants for quiver varieties related by a mutation are the same under a cluster transformation of variables

Ruan '17

$$\mathscr{F}_g(z) = \mathscr{F}_g(z')$$

Bonelli-Sciarappa-Tanzini-Vasko, '15

- The vortex partition function is precisely the quasimap $I$ -function in genus 0.

Webb '18

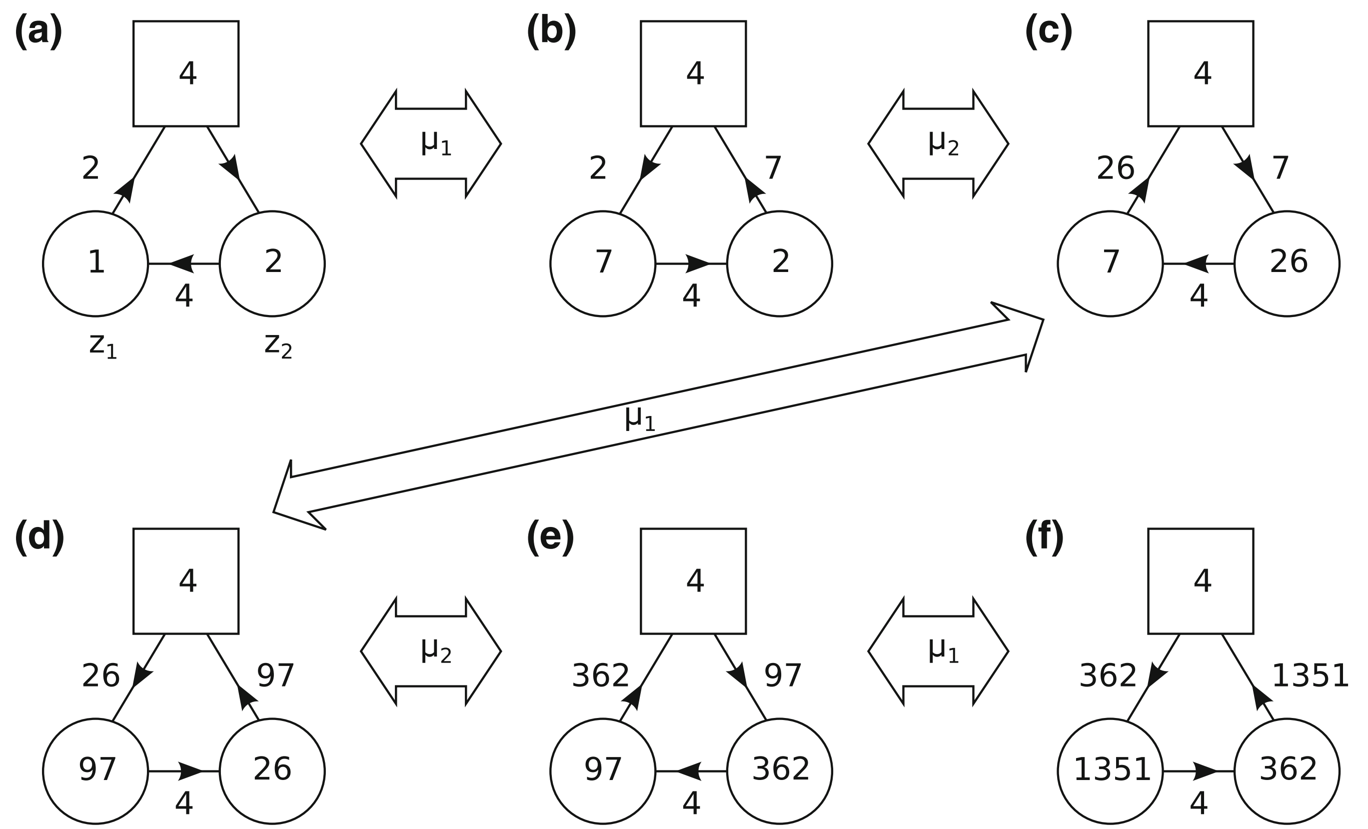
- For $A_n$ linear quivers corresponding to flag varieties, the conjecture has recently been proved

Zhang '21

# Examples

- There are many examples that can be studied…

Gulliksen-Negård CY

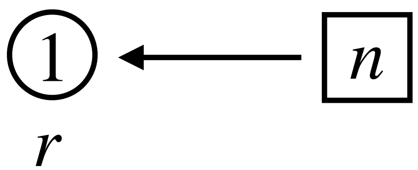



| | Mutations | Kähler coordinates | |
|---|---|---|---|
| (a) | · | $z_1$ | $z_2$ |
| (b) | $\mu_1$ | $z_1^{-1}$ | $z_2(1+z_1)^4$ |
| (c) | $\mu_2\mu_1$ | $z_1^{-1}\left(1+z_2(1+z_1)^4\right)^4$ | $z_2^{-1}(1+z_1)^{-4}$ |
| (d) | $\mu_1\mu_2\mu_1$ | $z_1\left(1+z_2(1+z_1)^4\right)^{-4}$ | $z_2^{-1}(1+z_1)^{-4}\left(1+z_1^{-1}\left(1+z_2(1+z_1)^4\right)^4\right)^4$ |

- Note: the duality breaks down at $z_i = -1$. This is actually a very interesting point. I will revisit this point later.

# Another conjecture on quantum cohomology

- Twisted chiral ring = quantum cohomology




$$\sigma^n = z$$

- Baxter polynomial = generator of the cohomology ring

$$Q(x) = \det(x - \sigma)$$

- For Grassmannian, $Q(x)$ classically generates the Chern classes of the tautological bundle

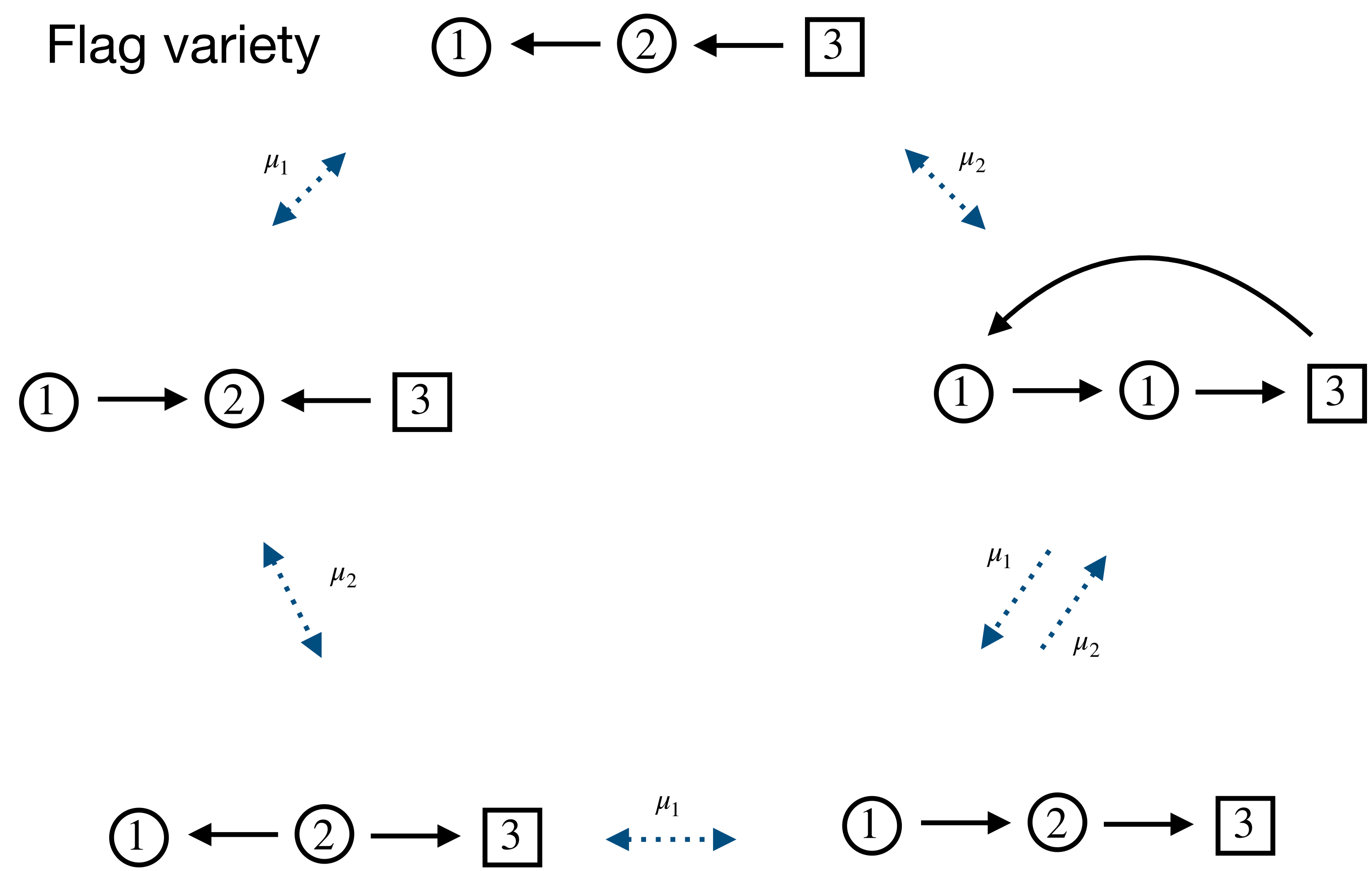
- Under duality, Baxter polynomials map as cluster variables!

$$Q_i(x) Q_i'(x) \sim \prod_{i \to j} Q_j(x)^{b_{ij}} + \prod_{j \to i} Q_j(x)^{-b_{ij}}.$$

- This hints at a deeper connection between quantum groups and quantum cohomology

# Positive GLSM quivers

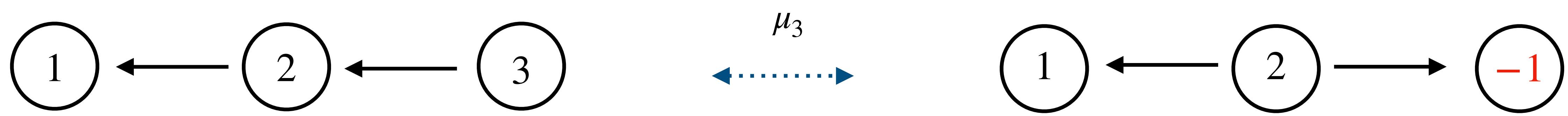
- We introduce the notion of **positive GLSM quivers**
- Recall that $N_i' = \max(N_i^{\text{in}}, N_i^{\text{out}}) - N_i$
- A quiver defining a GLSM is positive if all the gauge group ranks stay positive in *any* duality frame



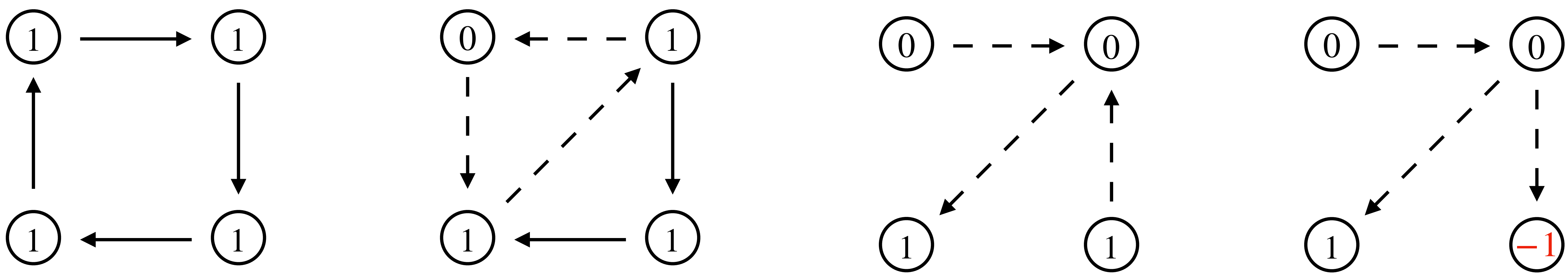

- If $N_i' < 0$, then there is no ground state and $Z_{\mathbb{S}^2} = 0$. We say supersymmetry is broken.
- If $Z_{\mathbb{S}^2} = 0$ in some duality frame, then it vanishes in **all** duality frames

# Unframed quivers

- Positivity is a very strong condition on unframed quivers
- Suppose we gauge the framed nodes in a flag variety



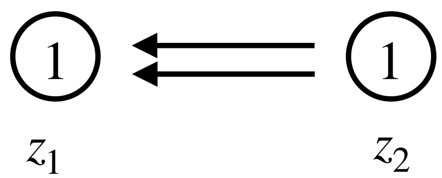

- In general, any quiver tail with single arrows will violate positivity in some mutation class




- The classification problem is still open
- This motivates us to study quivers with multiple arrows.

# Kronecker quiver

- The simplest positive quiver corresponds to affine $A_1$, also known as the Kronecker quiver



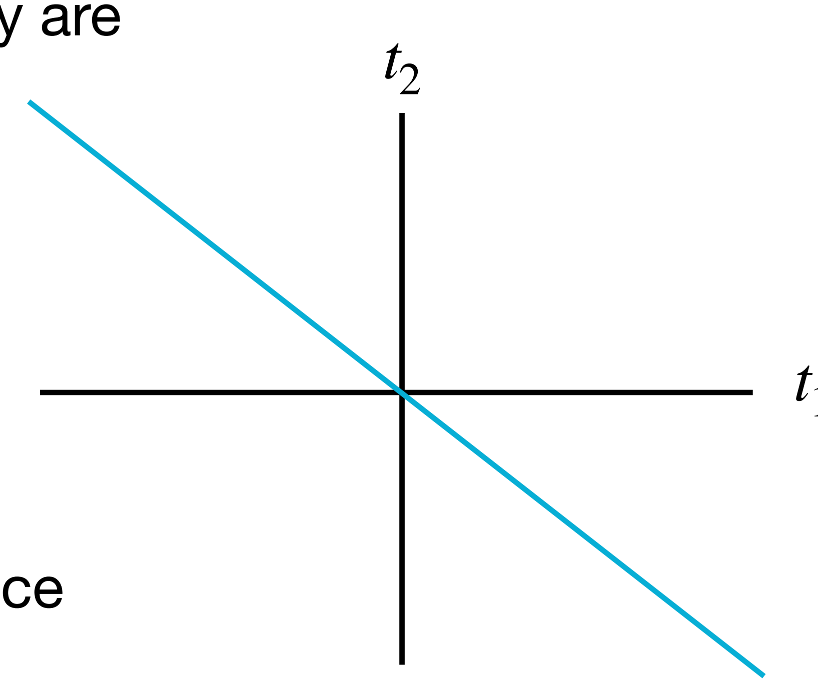

$$z_i \sim e^{-t_i}$$

| | $\phi_1$ | $\phi_2$ |
|---|---|---|
| $U(1)_1$ | 1 | 1 |
| $U(1)_2$ | -1 | -1 |

- Naively there are two independent coordinates $t_1$ and $t_2$, but they are actually constrained by the momentum map (D-term) equations

$$|\phi_1|^2 + |\phi_2|^2 = t_1$$

$$-|\phi_1|^2 - |\phi_2|^2 = t_2$$




- The actual phase space lies in a codim-1 locus of the 2-dim space

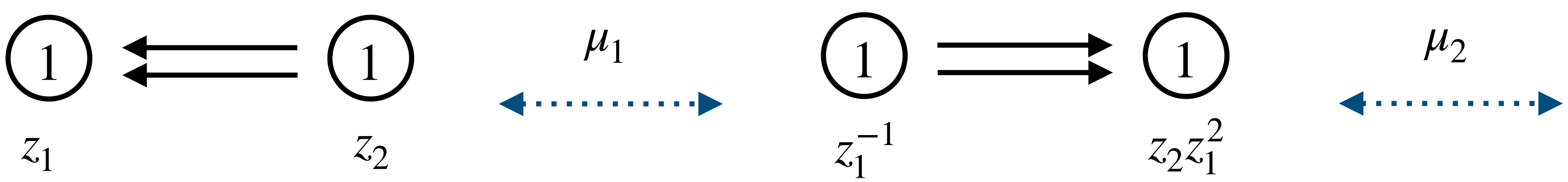
- The importance of having a redundant Kähler coordinate will become clear later

# Kronecker quiver - infinite duality chain

- Can also see from the quantum cohomology (twisted chiral ring) relations that $z_2 = z_1^{-1}$

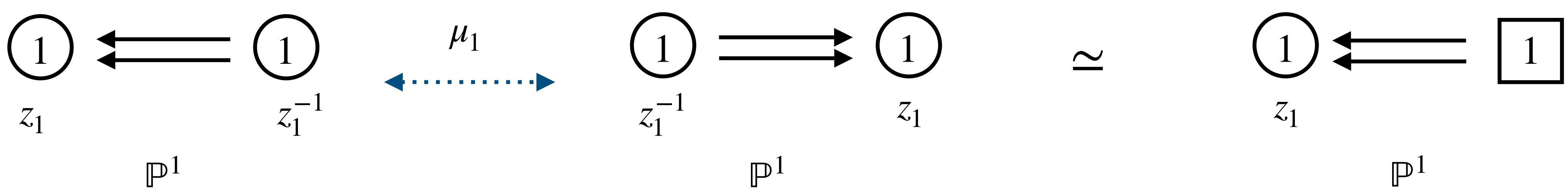
$$(\sigma_1 - \sigma_2)^2 - z_1 = 0 \qquad\qquad 1 - z_2(\sigma_2 - \sigma_1)^2 = 0$$

- The Kronecker quiver is of affine type, so one expects it to be infinite-mutation type



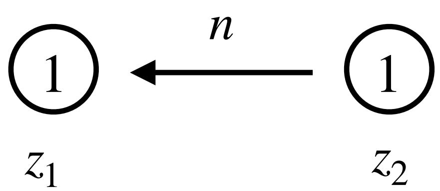

- But the GLSM constraint $z_2 = z_1^{-1}$ makes it finite

- Instead of an infinite class of equivalent GIT quotients we only have one:




- Question: Can all quivers without framing be realized as quivers with framing?

# n-Kronecker quiver



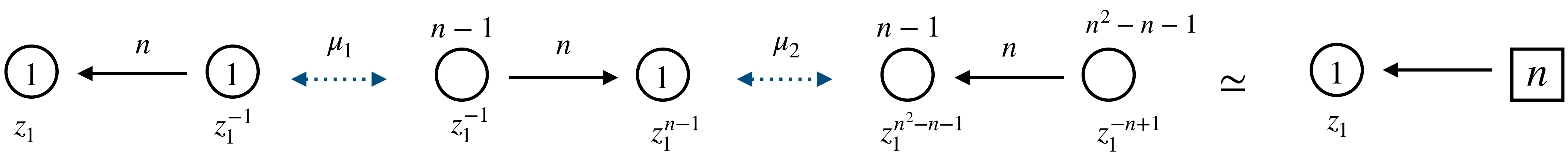

| | $\phi_1$ | $\phi_2$ | $\cdots$ | $\phi_n$ |
|---|---|---|---|---|
| $U(1)_1$ | 1 | 1 | $\cdots$ | 1 |
| $U(1)_2$ | -1 | -1 | $\cdots$ | -1 |

$$\sum_{i=1}^{n} |\phi_i|^2 = t_1 \,, \qquad -\sum_{i=1}^{n} |\phi_i|^2 = t_2$$

- By the same argument, we obtain the quantum cohomology of $\mathbb{P}^{n-1}$

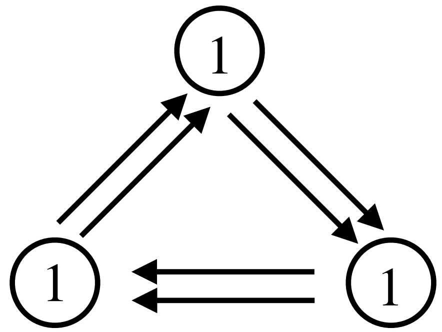
$$(\sigma_1 - \sigma_2)^n - z_1 = 0 \,, \qquad 1 - z_2(\sigma_1 - \sigma_2)^n = 0$$

- We still have $z_2 = z_1^{-1}$, but we get an infinite class of equivalent GIT quotients of $\mathbb{P}^n$



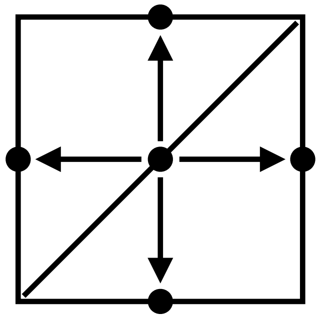

- Abelian - nonabelian duality
- Can we also get an infinite class of Calabi-Yau spaces from unframed quivers?

# Markov quiver




| | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ |
|---|---|---|---|---|---|---|
| $U(1)_1$ | 1 | 1 | -1 | -1 | 0 | 0 |
| $U(1)_2$ | 0 | 0 | 1 | 1 | -1 | -1 |
| $U(1)_3$ | -1 | -1 | 0 | 0 | 1 | 1 |

- Simplest positive quiver with 3 nodes. Also arises from the ideal triangulation of a once-punctured torus

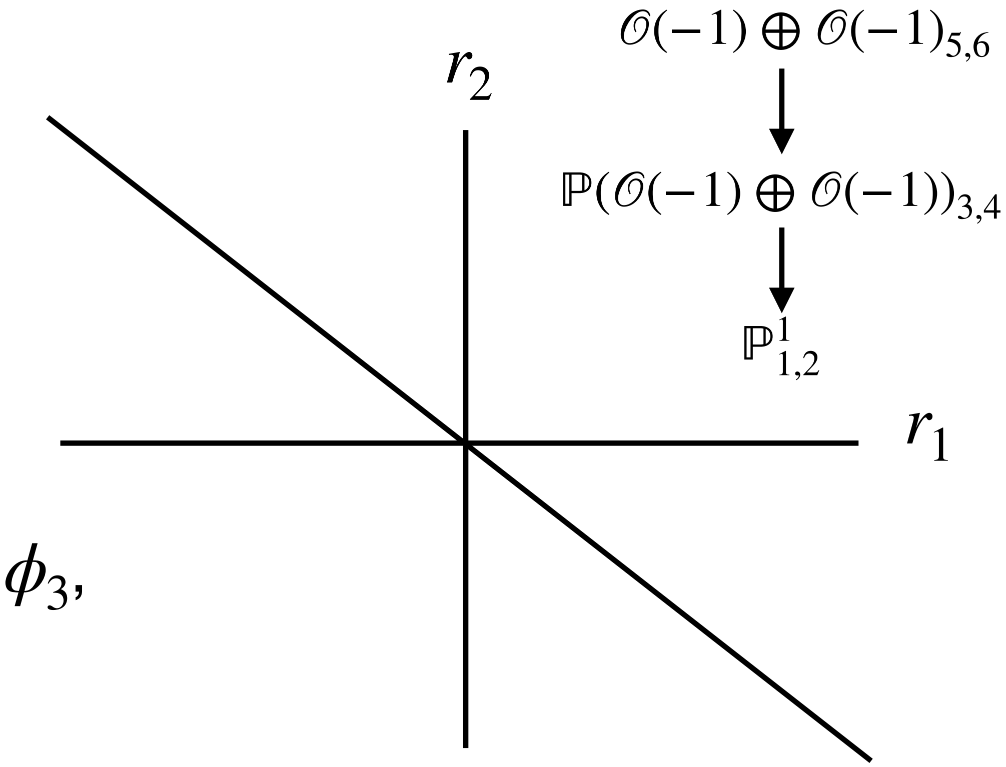


$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 = r_1$$
$$|\phi_3|^2 + |\phi_4|^2 - |\phi_5|^2 - |\phi_6|^2 = r_2$$
$$|\phi_5|^2 + |\phi_6|^2 - |\phi_1|^2 - |\phi_2|^2 = r_3$$

- Consistency of the equations imply $t_1 + t_2 + t_3 = 0$ and we may again decouple an overall $U(1)_+ \subset U(1)_1 \times U(1)_2 \times U(1)_3$
- Calabi-Yau condition: # incoming arrows = # outgoing
- Let us examine the phase space in detail

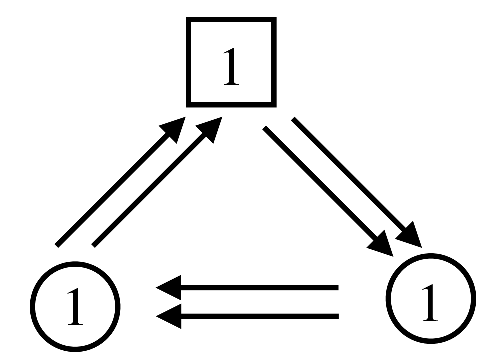
# Markov quiver - phase space

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 = r_1$$
$$|\phi_3|^2 + |\phi_4|^2 - |\phi_5|^2 - |\phi_6|^2 = r_2$$
$$|\phi_5|^2 + |\phi_6|^2 - |\phi_1|^2 - |\phi_2|^2 = -r_1 - r_2$$



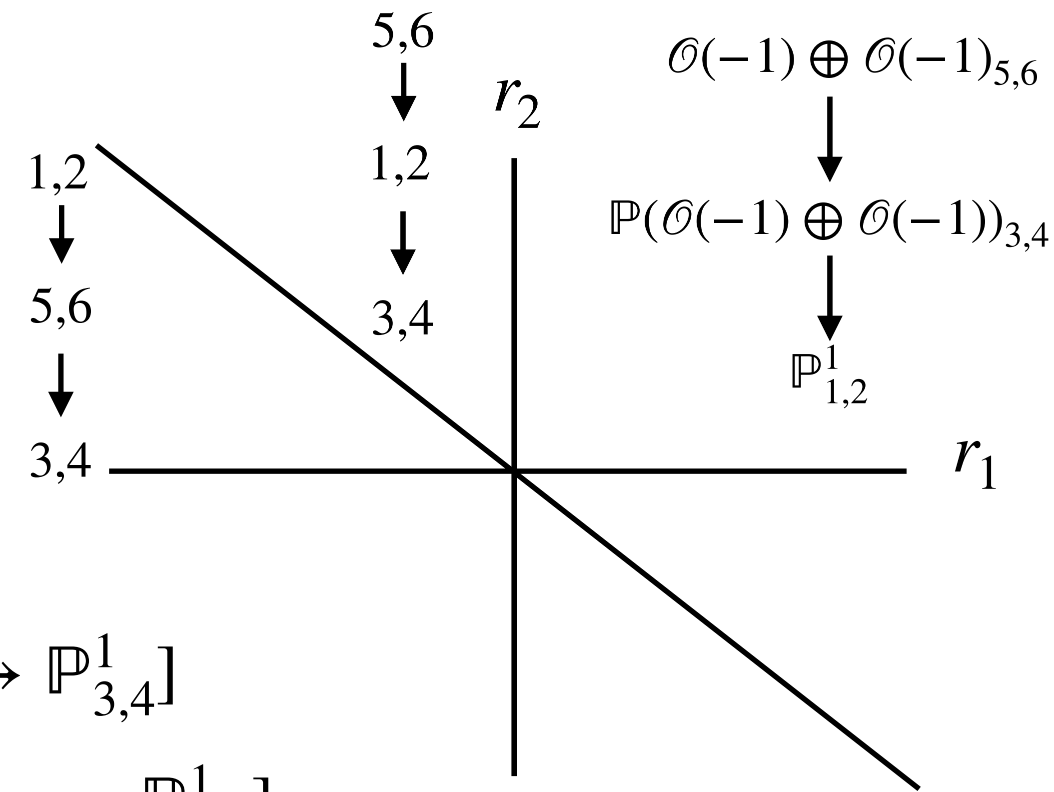

- $r_1 \gg 0$: $\phi_1$, $\phi_2$ cannot all vanish, parametrize a $\mathbb{P}^1$ base. $\phi_3$, $\phi_4$ describe the fiber directions. Thus the first equation describes the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, the resolved conifold.
- $r_2 \gg 0$: $\phi_3$, $\phi_4$ cannot all vanish. Gauging $U(1)_2$ will give a projectivization of the fiber from the $r_1 \gg 0$ equation, namely $\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$. Now $\phi_5$, $\phi_6$ define another fiber growing on top of it.
- This is consistent with the third equation, so in the $r_1, r_2 \gg 0$ phase, it engineers the following geometry

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1)_{5,6} \to \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))_{3,4} \to \mathbb{P}^1_{1,2}$$



# Markov quiver - flop transitions

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 = r_1$$
$$|\phi_3|^2 + |\phi_4|^2 - |\phi_5|^2 - |\phi_6|^2 = r_2$$
$$|\phi_5|^2 + |\phi_6|^2 - |\phi_1|^2 - |\phi_2|^2 = -r_1 - r_2$$




- Now study the region $r_1 \ll 0$ and $r_2 \gg 0$
- The first equation defines Tot$[\mathcal{O}(-1) \oplus \mathcal{O}(-1))_{1,2} \to \mathbb{P}^1_{3,4}]$
- The second equation defines Tot$[\mathcal{O}(-1) \oplus \mathcal{O}(-1))_{5,6} \to \mathbb{P}^1_{3,4}]$
- Same base, different fibers. Now examine the third equation
- If $r_1 + r_2 > 0$, then $\phi_1, \phi_2$ cannot all vanish, and $\phi_5, \phi_6$ are fiber directions

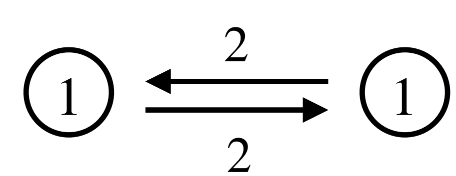
  $$\mathcal{O}(-1) \oplus \mathcal{O}(-1)_{5,6} \to \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))_{1,2} \to \mathbb{P}^1_{3,4}$$

  If $r_1 + r_2 < 0$, then fiber and base are again exchanged

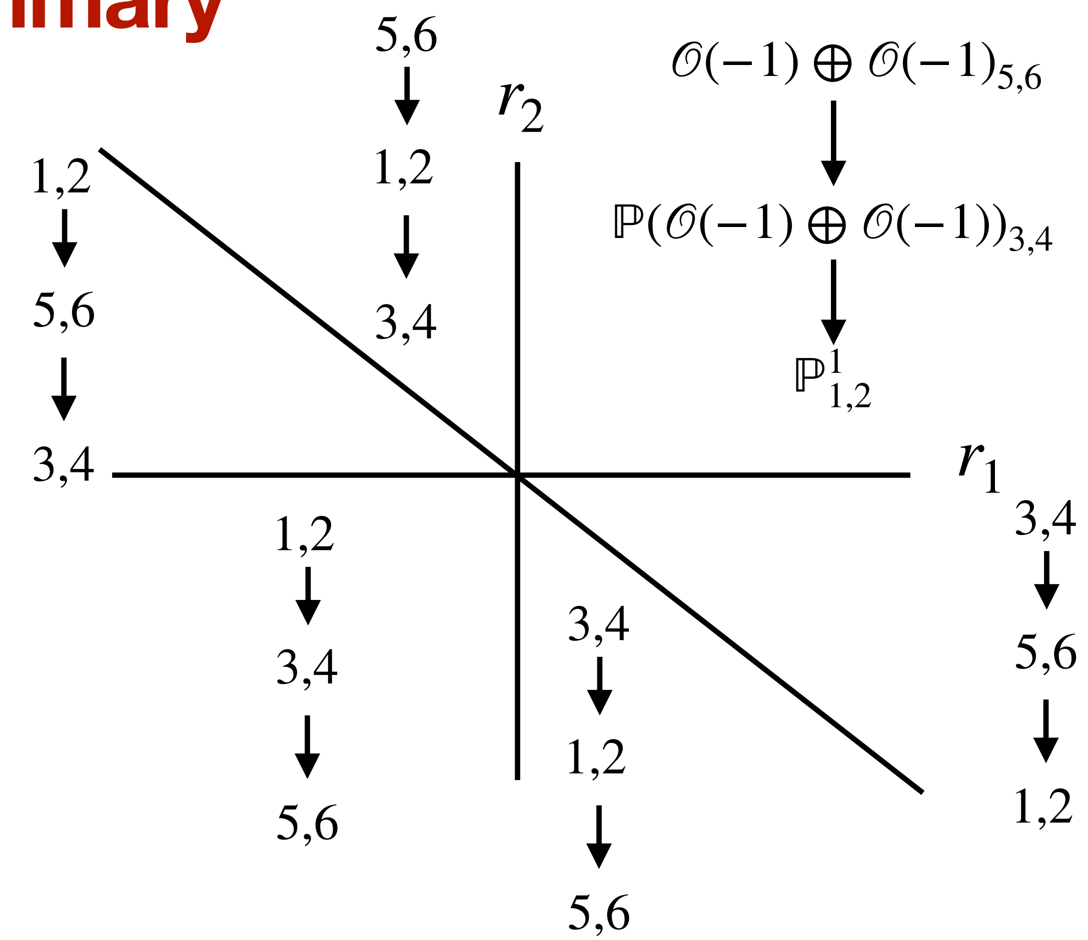
  $$\mathcal{O}(-1) \oplus \mathcal{O}(-1)_{1,2} \to \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))_{5,6} \to \mathbb{P}^1_{3,4}$$

- If $r_1 + r_2 = 0$, then $\phi_1, \phi_2$ and $\phi_5, \phi_6$ are identified

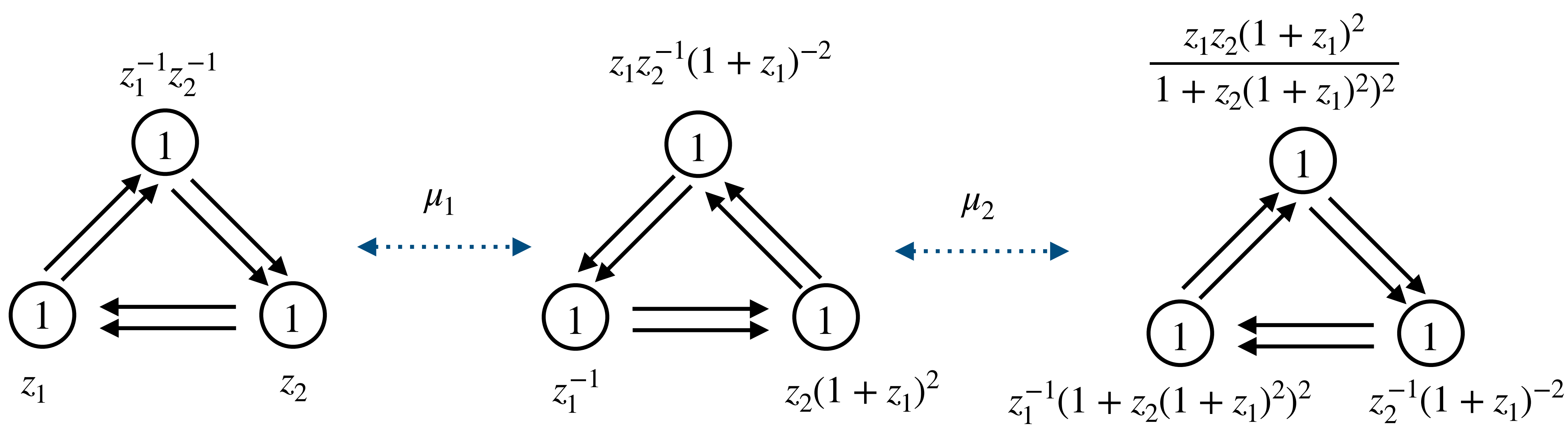
  $$\mathcal{O}(-1) \oplus \mathcal{O}(-1)_{1,2} \to \mathbb{P}^1_{3,4}$$



# Markov quiver - summary

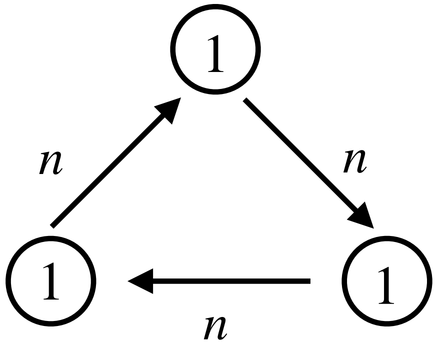



There are 3!=6 phases, related by flop transitions between fiber and base




Tessellates the Kähler moduli space moduli space

# n-Markov quiver




- The n-Markov quiver is particularly interesting because of its relation to an nonabelian theory

- Twisted chiral ring relations

$$(\sigma_i - \sigma_{i-1})^2 - z_i(\sigma_i - \sigma_{i+1})^2 = 0, \qquad i = 1, 2, 3$$

- We decouple the overall $U(1)$ by the constraints $z_1 z_2 z_3 = 1$ and $\sigma_{3,1} = -\sigma_{1,2} - \sigma_{2,3}$, where $\sigma_{i,j} \equiv \sigma_i - \sigma_j$. Integrating out these bifundamentals requires that $\sigma_i \neq \sigma_j$ and therefore we can rewrite the chiral ring relations as

$$\left(\frac{\sigma_{1,2}}{-\sigma_{1,2} - \sigma_{2,3}}\right)^n = z_1, \qquad \left(\frac{\sigma_{2,3}}{-\sigma_{1,2} - \sigma_{2,3}}\right)^n = z_2\,.$$

- Apart from the decoupled $U(1)$ direction, there is no supersymmetric vacuum at generic points on the Kähler moduli space. We observe that at the origin $(t_1, t_2) = (0,0)$, the twisted F-term equations take the same form as the SU(3) theory with $n$ massless chiral multiplets.

# 2d SQCD


- Twisted chiral ring relations

$$\left(\frac{\sigma_i}{-\sigma_1 - \sigma_2 - \cdots \sigma_{k-1}}\right)^n = 1, \qquad i = 1,\ldots,k-1$$

- If $n$ is a multiple of 3, then a one-dimension non-compact Coulomb branch appear in the direction $(\sigma_1, \sigma_2, \sigma_3) = (1, e^{2\pi i/3}, e^{4\pi i/3})\sigma$

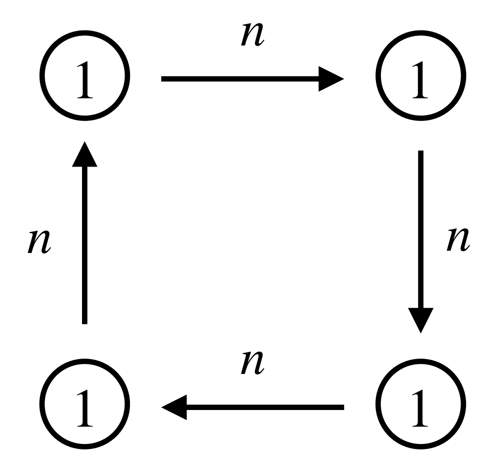
- An overall scaling will not change the complex direction, and solutions related by permutations are identified under the Weyl group. It was proposed that singularities correspond to $k$ distinct $n$-th roots of unity, modulo overall scaling, that sum to zero.

- For higher rank, multiple Coulomb branch directions may open up. For $(k, n) = (4,8)$, there are two non-compact directions along $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (1, -1, 1, -1)\sigma$ and $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (1, -1, 1, -1)\sigma$

Hori-Tong '06

# Abelian necklace quivers

- What about SU($k$) SQCD? The answer is given by necklace quivers




- Twisted chiral ring relations

$$\left(\frac{\sigma_{i-1}-\sigma_i}{\sigma_i-\sigma_{i+1}}\right)^n = z_i\,, \qquad i=1,\ldots,k$$

- We factor out the overall U(1) by the constraints $z_1 z_2 \cdots z_k = 1$ and

$$\sigma_k - \sigma_1 = -\sum_{i=1}^{k-1}(\sigma_i - \sigma_{i+1})\,.$$

- If $n$ is a multiple of 3, then a pair of one-dimension non-compact Coulomb branches appear in the directions $(\sigma_1, \sigma_2, \sigma_3) = (1, e^{2\pi i/3}, e^{4\pi i/3})\sigma$ and $(1, e^{4\pi i/3}, e^{2\pi i/3})\sigma$

- For nonabelian groups, Cartan elements related by Weyl symmetry are identified. Such Weyl symmetry does not appear naturally in the abelian quiver. So we find many more solutions.

# Quantum Coulomb branches

| $k\backslash n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 2 | 0 |
| 5 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| 6 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |

The number of quantum Coulomb branches of massless $SU(k)$ SQCD with $n$ chiral multiplets.

| $k\backslash n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 |
| 4 | 3 | 0 | 9 | 0 | 15 | 0 | 21 | 0 | 27 | 0 |
| 5 | 0 | 0 | 0 | 24 | 60 | 0 | 0 | 0 | 24 | 0 |
| 6 | 10 | 30 | 100 | 0 | 340 | 0 | 640 | 270 | 1090 | 0 |

The number of quantum Coulomb branches of the abelian $n$-necklace quiver with $k$ nodes

# Discrete $\theta$-angle

- The foregoing analysis is a minor modification of Hori and Tong's analysis of SU($k$) SQCD. But we now have the additional freedom to tune the Kähler parameters.

- When $k = 3$, we find that no additional singularity arises and the origin is the only singular point. The point where the discrete $\theta$-angle is turned on, $\theta = \pi$, is regular for any $n$ so is a smooth point on the moduli space.

| $k \backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 2 | 5 | 4 | 9 | 6 | 13 |
| 5 | 0 | 0 | 0 | 0 | 0 | 12 | 0 |
| 6 | 1 | 0 | 31 | 0 | 109 | 24 | 235 |

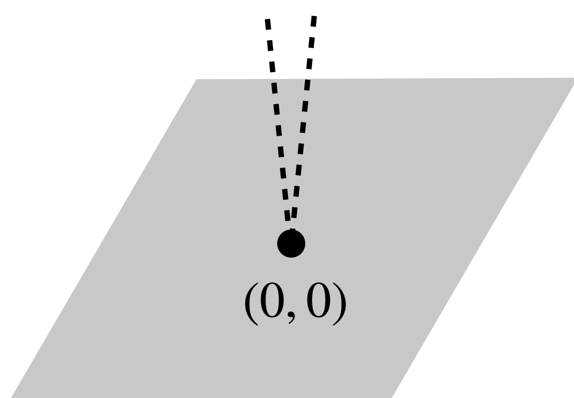
: The number of quantum Coulomb branches of the abelian $n$-necklace quiver with $k$ nodes at singularity of the Kähler moduli space, $t_i = i\pi$ for $i = 1, \ldots, k-1$.

- This is also the point $z = e^{2\pi r + i\theta} = -1$ where duality fails.

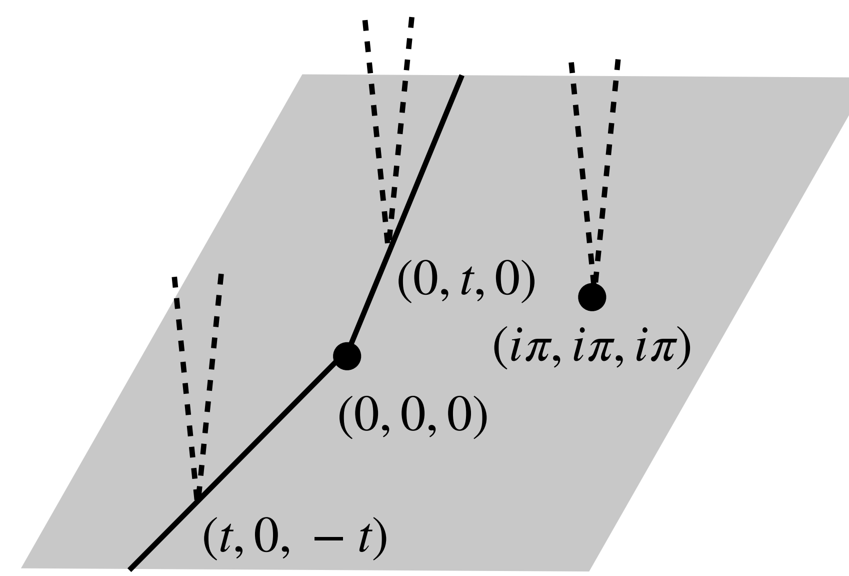
# New singularities on the Kähler moduli space

- We find a new feature when $k > 3$. There is a continuous family of solutions that support quantum Coulomb branches as we tune the Kähler parameters.

Kähler moduli space




$(k, n) = (3,3)$




$(k, n) = (4,2)$

## Summary

- Defined the notion of positive GLSM quiver, using dualities as cluster transformation.
- Identified Kronecker and Markov quivers as the simplest examples -> infinitely many equivalent geometries.
- Found abelian necklace quiver to realize features of nonabelian 2d SQCD, and found new quantum Coulomb branches on the Kähler moduli space.

## Future directions

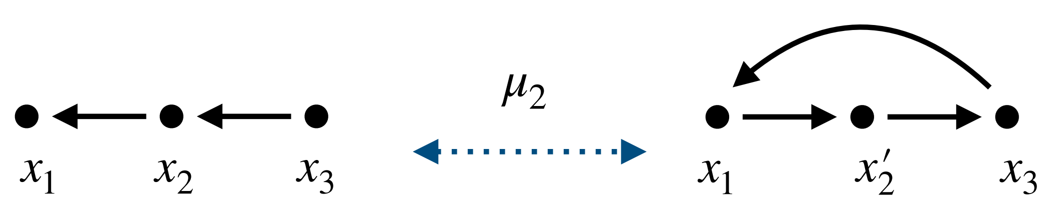
- A general theory for unframed quivers seems difficult. We need to study case by case.
- We only studied abelian examples. Much more can be studied for nonabelian cases.
- The n-Markov quiver is regular at $\theta = \pi$. Connection to SU(3) SQCD at $\theta = \pi$?
- Study singular conformal field theories on the quantum Coulomb branches.

# Thank you for your time!

# Appendix

# Cluster algebra (of geometric type)

- Commutative ring with a distinguished set of generators called cluster variables $x_i$
- Other generators are defined recursively by
  - Quiver: directed graph with skew-symmetric adjacency matrix $b_{ij}$
  - Mutation $\mu_k : (b_{ij}, x_i) \to (b'_{ij}, x'_i)$

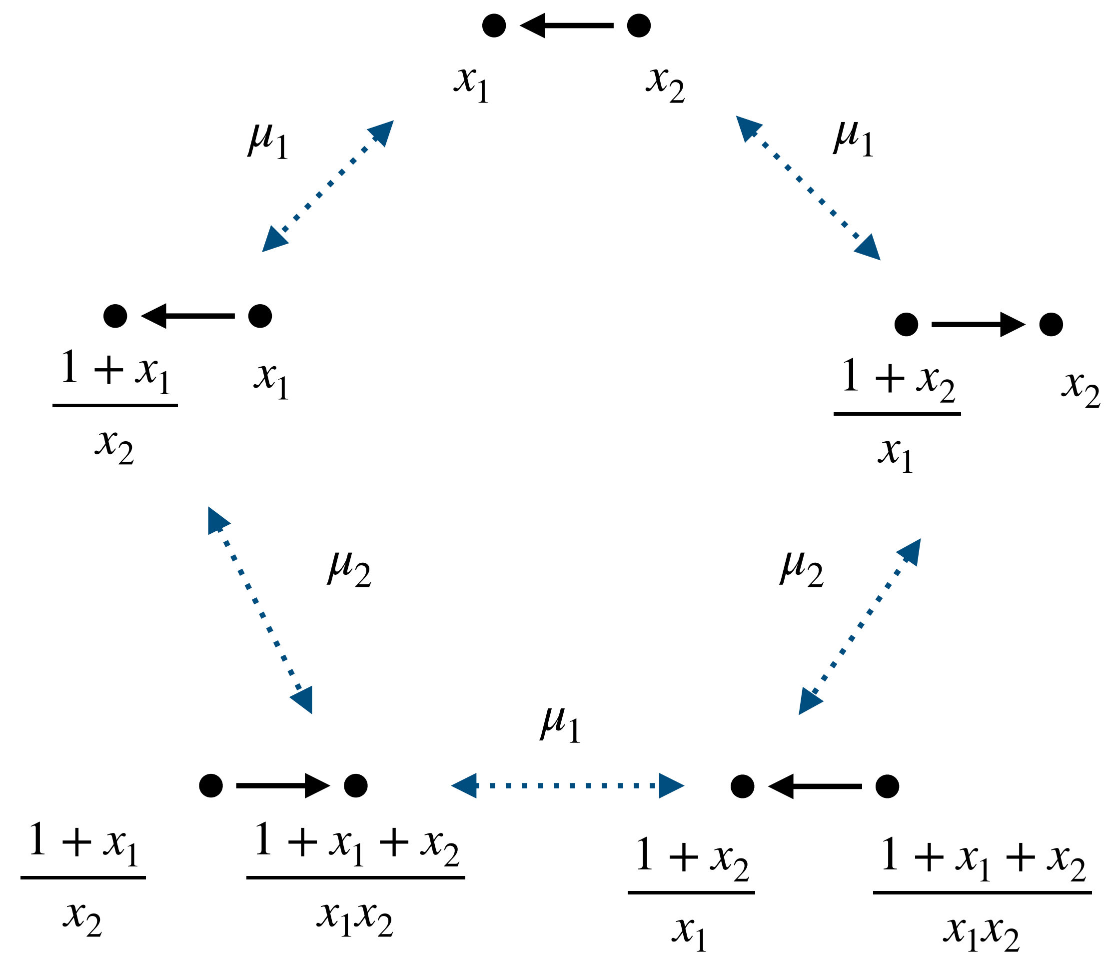



$$x_k\, x'_k = \prod_{j \leftarrow k} x_j^{b_{kj}} + \prod_{j \to k} x_j^{b_{jk}}$$

Fomin-Zelevinsky '01

$A_2$ example

$$b_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



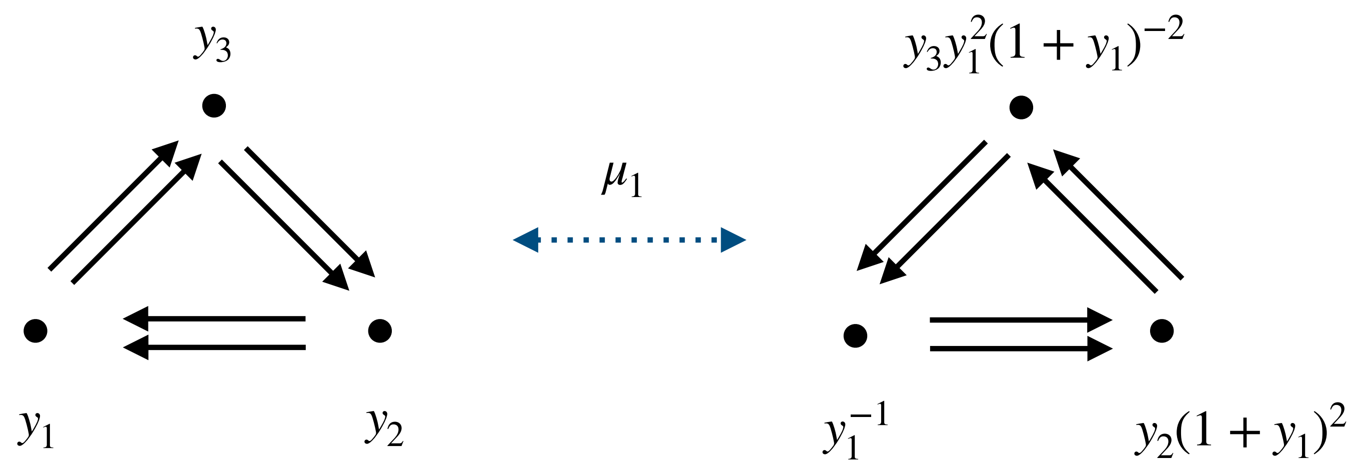

$A_2$ cluster algebra is generated by $\left\{ x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1 x_2} \right\}$

- More general definition includes coefficient variables $y_i$

# Examples of cluster algebras

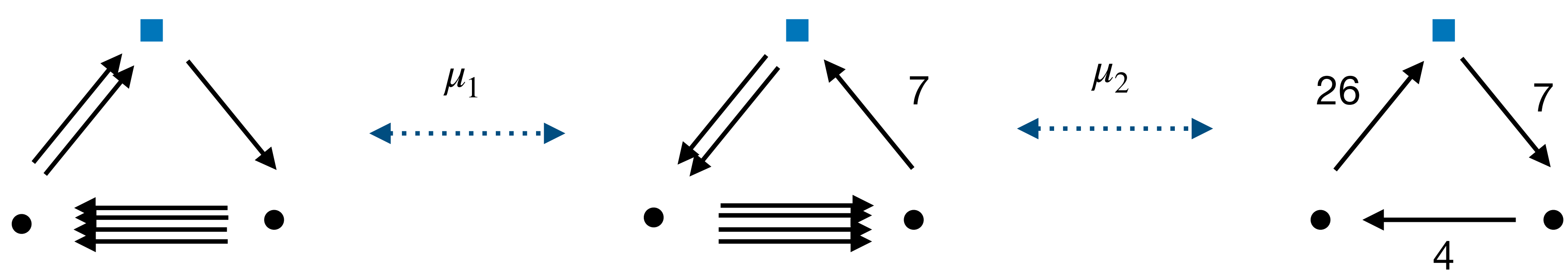
- The coefficient variables mutate as $y'_k = y_k^{-1}, \quad y'_i = y_i y_k^{[b_{ki}]_+}(1 + y_k)^{-b_{ki}}$

Markov quiver




Gulliksen-Negård quiver

We allow frozen nodes that do not mutate




- Generically, mutation generates an infinite class of quivers or cluster variables
- A quiver is **mutation-finite** if its mutation equivalence class is finite
- A cluster algebra is of **finite type** if there are finite number of variables
- When are cluster algebras finite?

# Key properties

- Finite-type iff the graph of the quiver is a finite-type Dynkin diagram

Fomin-Zelevinsky '03

- Poisson structure $\{y_i\,, y_j\} := b_{ij} y_i y_j$ is mutation invariant
- Cluster mutations are canonical transformations on the Kähler moduli