Virtual Cycles on Projective Completions

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1. Noncomact spaces.

- Invariants are obtained by integrations over moduli spaces. For instance Gromov-Witten invariants are integrations over the moduli spaces of stable maps. Donaldson-Thomas invariants are integrations over the moduli space of stable sheaves.
- Integrations are well-defined when the moduli spaces are compact and have nice enough deformation-obstruction theories, so called *perfect obstruction theory* (Li-Tian, Behrend-Fantechi).
- When the moduli space is not compact, but is equipped with the perfect obstruction theory, the integrand is well-defined, but the integrations are not well-defined.

- An important principle of (several) localisations is to integrate over smaller spaces rather than original spaces.
- Torus localisation (Graber-Pandharipande) localises the integrand to the fixed loci of the torus action on the moduli space.
- Cosection localisation (Kiem-Li) localised it to the degeneracy loci of the cosection.
- So even if the moduli space is noncompact, we can do integrations so long as we can localise them to a compact subspace.

- There are several interesting noncompact moduli spaces having perfect obstruction theories.
- A moduli space of stable maps to a GIT quotient with p-fields can be considered as a *liner object* to study a moduli space of stable maps to a complete intersection into the GIT quotient, which is *quite non-linear*.
- A moduli space of stable sheaves on a surface with Higgs fields defines Vafa-Witten theory.

- These examples are cones over the compact moduli spaces. A moduli space of stable maps to a GIT quotient with p-fields projects to the space without p-fields. A moduli space of stable sheaves on a surface with Higgs fields also projects to the space without fields.
- So these have fiberwise $\mathbb{C}^*$-actions fixing the vertex spaces which are compact. We can define invariants.
- For these two examples, $\mathbb{C}^*$-actions define cosections degenerate on the vertex spaces as well.
- It is known that two localisations give different invariants.

2. Qunatum Lefschetz.

- For simplicity, we fix our GIT quotient $\mathbb{P}^4 = \mathbb{C}^5 /\!/ \mathbb{C}^*$. We also fix integers g and d.
- We denote $M(\mathbb{P}^4)$ by the moduli space of stable maps to $\mathbb{P}^4$ of genus g and degree d.
- (Ignoring stability condition), an element of $M(\mathbb{P}^4)$ is

$$(C, L, u)$$

where C is a genus g curve, L is a degree d bundle on C, $u : \mathcal{O}_C \to L^{\oplus 5}$ is a section.
- The virtual cycle (or the integrand) $[M(\mathbb{P}^4)]^{vir}$ is defined as a homology class of degree $2(5d + 1 - g)$.

- Let $X \subset \mathbb{P}^4$ be a smooth quintic threefold defined by the defining equation $f \in \Gamma(\mathcal{O}_{\mathbb{P}^4}(5))$.
- We denote $M(X)$ by the moduli space of stable maps to X of genus g and degree d.
- An element of $M(X)$ is

$$(C, L, u) \;\text{ such that }\; f(u) = 0 \in \Gamma(L^{\otimes 5}),$$

  where $(C, L, u) \in M(\mathbb{P}^4)$.
- The virtual cycle $[M(X)]^{vir}$ is defined as a homology class of degree 0. Since $M(X)$ is a closed subspace of $M(\mathbb{P}^4)$, $[M(X)]^{vir}$ can be considered as a homology class of $M(\mathbb{P}^4)$.

- Now we denote $M(\mathbb{P}^4)^p$ by the moduli space of stable maps to $\mathbb{P}^4$ with the p-fields.
- An element of $M(\mathbb{P}^4)^p$ is

$$(C, L, u, p) \text{ with } p \in \Gamma(L^{\otimes -5} \otimes \omega_C)$$

  where $(C, L, u) \in M(\mathbb{P}^4)$.
- There is a projection map $M(\mathbb{P}^4)^p \to M(\mathbb{P}^4)$ forgetting p. Also there is an inclusion map $M(\mathbb{P}^4) \hookrightarrow M(\mathbb{P}^4)^p$ setting $p = 0$.
- The virtual cycle $[M(\mathbb{P}^4)^p]^{vir}$ is defined as a homology class of degree 0.

- There are two ways to localise $[M(\mathbb{P}^4)^p]^{vir}$ to $M(\mathbb{P}^4)$ – torus and cosection localisatioins.
- Let $E_2 := R\pi_* L^{\otimes 5}[1]$ be the complex in the derived category of $M(\mathbb{P}^4)$, where $\pi : C \to M(\mathbb{P}^4)$ is the universal curve.
- Then the torus localised class $[M(\mathbb{P}^4)^p]^{vir}_{\mathsf{T}}$ of $[M(\mathbb{P}^4)^p]^{vir}$ is equal to

$$\left[c(E_2[-1]) \cap [M(\mathbb{P}^4)]^{vir}\right]_{\deg=0}.$$

- Whereas the cosection localised class $[M(\mathbb{P}^4)^p]^{vir}_{\sigma}$ of $[M(\mathbb{P}^4)^p]^{vir}$ is equal to $[M(X)]^{vir}$ (Chang–J.Li, Kim–O., Chang–M.-L.Li, Chen–Janda–Webb, Picciotto).

- In genus 0, $M(\mathbb{P}^4)$ is smooth so that $[M(\mathbb{P}^4)]^{vir} = [M(\mathbb{P}^4)]$ and $E_2[-1]$ is a vector bundle, hence

$$[M(\mathbb{P}^4)^p]^{vir}_{\mathsf{T}} \;=\; e(E_2[-1]) \cap [M(\mathbb{P}^4)].$$

  In this case, $[M(\mathbb{P}^4)^p]^{vir}_{\mathsf{T}} = [M(\mathbb{P}^4)^p]^{vir}_{\sigma}$ (Kim-Kresch-Pantev). This is called the *quantum Lefschetz property*.
- In genus > 0, $[M(\mathbb{P}^4)^p]^{vir}_{\mathsf{T}} \neq [M(\mathbb{P}^4)^p]^{vir}_{\sigma}$ (Givental).

3. Jiang-Thomas.

- Let S be a surface and $c \in H^{even}(S)$ be a cohomology class.
- We denote $M(S)$ by the moduli space of stable sheaves F on S such that $ch(F) = c$.
- The virtual cycle $[M(S)]^{vir}$ is defined as a homology class of degree $2(1 - \chi(F, F))$.

- We denote $M(K_S)$ by the moduli space of

$$(F, s),\ \mathcal{F} \in M(S),\ s \in \mathrm{Hom}(F, F \otimes K_S)$$

- There is a projection map $M(K_S) \to M(S)$ forgetting s. Also there is an inclusion map $M(S) \hookrightarrow M(K_S)$ setting $s = 0$.
- The virtual cycle $[M(K_S)]^{vir}$ is defined as a homology class of degree 0.

- There are two ways to localise $[M(K_S)]^{vir}$ to $M(S)$ – torus and cosection localisatioins.
- Let $E_2 := \tau^{[1,2]}\left(R\pi_*\mathcal{H}om(\mathcal{F},\mathcal{F})\right)[2]$ be the complex in the derived category of $M(S)$, where $\mathcal{F}$ is the (twisted) universal sheaf on $S \times M(S)$ and $\pi : S \times M(S) \to M(S)$ is the projection morphism.
- Then the torus localised class $[M(K_S)]^{vir}_{\mathsf{T}}$ of $[M(K_S)]^{vir}$ is equal to

$$\left[c(E_2[-1]) \cap [M(S)]^{vir}\right]_{\deg=0}.$$

- Letting $[M(K_S)]^{vir}_\sigma$ be the cosection localised class of $[M(K_S)]^{vir}$, Jiang-Thomas proved $[M(K_S)]^{vir}_{\mathsf{T}} \neq [M(K_S)]^{vir}_\sigma$ in general.
- More generally, let M be a quasi-smooth projective derived scheme and N be the (-1)-shifted cotangent bundle. Then we have two localisations $[N]^{vir}_{\mathsf{T}}$ and $[N]^{vir}_\sigma$ to M. Jiang-Thomas proved they may be different.

4. Projective completion.

- We denote by $M := M(\mathbb{P}^4)$ or $M(S)$ or a quasi-smooth projective derived scheme, and by $N := M(\mathbb{P}^4)^p$ or $M(K_S)$ or the (-1)-shifted cotangent bundle of M.
- We denote by $p : \mathbb{P}(N) \to M$ the projectivisation.
- Our purpose is to construct a reduced cycle $[\mathbb{P}(N)]^{red}$ of degree 0 satisfying

$$[N]^{vir}_{\mathsf{T}} - [N]^{vir}_{\sigma} \;=\; p_*[\mathbb{P}(N)]^{red}.$$

- In particular, if $N = M$, then $\mathbb{P}(N) = \varnothing$, hence

$$[N]^{vir}_{\mathsf{T}} \;=\; [N]^{vir}_{\sigma} \;=\; e(E_2) \cap [M]^{vir}.$$

- We prove it in a universal framework.

Virtual cycle

- Let M be a finite type DM stack.
- A complex E of vector bundles on M degrees in $0, -1$ with the morphism $\phi : E \to \mathbb{L}_M$ to the (truncated) cotangent complex in the derived category is a *perfect obstruction theory* if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is a surjection.
- A perfect obstruction theory gives rise to the virtual cycle $[M]^{vir}$ of degree $2\mathrm{rank}E$ (Behrend-Fantechi).

Cosection

- Let (M, E) be as before.
- A *cosection* is a morphism $\sigma : E^\vee \to \mathcal{O}_M[-1]$, and its *degeneracy locus* $M(\sigma)$ is a closed subspace of M defined by the ideal generated by the image of $h^1(\sigma)$.
- Then the localised virtual cycle $[M]^{vir}_\sigma$ is defined in the homology of $M(\sigma)$ whose pushforward to M is $[M]^{vir}$ (Kiem-Li).

Reduced cycle

- Let (M, E) be as before. Let L be a line bundle on M.
- If there is a morphism $E^\vee \to L[-1]$ surjective in h^1, then we can construct the reduced virtual cycle $[M]^{red}$ of degree $2\text{rank}E + 2$ such that

$$e(L) \cap [M]^{red} \;=\; [M]^{vir}$$

(Kiem-Li).

Set-up

- Let M be a finite type DM stack over a smooth Artin stack A with the relative perfect obstruction theory E_1. (This induces the absolute perfect obstruction theory.)
- Let E_2 be a 2-term complex of vector bundles on M degrees in $0, -1$.
- Let N be $h^0(E_2^\vee)$ as a space. Then E_2 on N is the relative perfect obstruction theory of N over M.
- Assume there exists a morphism $E_2[-1] \to E_1$ in the derived category whose cone, denoted by E, is a relative perfect obstruction theory of N over A.
- Assume for convenience that rank$E = -\dim A$.

- For $(M, N) = (M(\mathbb{P}^4), M(\mathbb{P}^4)^p)$, $E_1 = (R\pi_* L^{\oplus 5})^\vee$ and $E_2 = R\pi_* L^{\otimes 5}[1]$.
- The morphism $E_2[-1] \to E_1$ is zero.
- For $(M, N) = (M(S), M(K_S))$, $E_1^\vee[1] = E_2 = \tau^{[1,2]}(R\pi_* \mathcal{H}om(\mathcal{F}, \mathcal{F}))[2]$.
- The morphism $E_2[-1] \to E_1$ is induced by the derived structure.

- Assume further that there is a morphism $E_1^\vee \to E_2[-1]$.
- For $(M,N) = (M(\mathbb{P}^4), M(\mathbb{P}^4)^p)$, this is induced by the defining equation $f : \mathbb{C}^5 \to \mathbb{C}$.
- For $(M,N) = (M(S), M(K_S))$, this is the identity morphism.
- There is the tautological morphism $E_2 \to \mathcal{O}_N$ induced by the fiberwise $\mathbb{C}^*$-action on N. This induces $E_1^\vee \to \mathcal{O}_N[-1]$.
- Assume this again induces $\sigma : E^\vee \to \mathcal{O}_N[-1]$.

- With this set-up (N, E) defines $[N]^{vir}$.
- Fiberwise $\mathbb{C}^*$-action defines $[N]^{vir}_{\mathsf{T}}$.
- σ defines $[N]^{vir}_{\sigma}$.

Theorem

$$[N]^{vir}_{\mathsf{T}} - [N]^{vir}_{\sigma} \;=\; p_*[\mathbb{P}(N)]^{red}.$$

Proof

- Take the projective completion $\pi : \overline{N} := \mathbb{P}(N \oplus \mathcal{O}_M) \to M$.
- Extend E to $\overline{E}$ on $\overline{N}$ to be a relative perfect obsturction theory of A.
- Then we show $\pi_*[\overline{N}]^{vir} = [N]^{vir}_{\mathsf{T}}$ by $\mathbb{C}^*$-localisation.
- Extend σ to $\overline{\sigma} : \overline{E}^{\vee} \to \mathcal{O}_{\overline{N}}(\mathbb{P}(N))[-1]$ on $\overline{N}$. Using this we compute

$$\pi_*[\overline{N}]^{vir} = [N]^{vir}_{\sigma} + p_*[\mathbb{P}(N)]^{red}.$$

Extension $\overline{E}$

- The projective completion $\overline{N}$ is defined to be the quotient

$$q : N \times \mathbb{C} \longrightarrow (N \times \mathbb{C})//\mathbb{C}^*.$$

- Here $E \oplus \mathcal{O}_N$ on $N \times \mathbb{C}$ gives rise to a perfect obstruction theory

$$E \oplus \mathcal{O}_N \longrightarrow \mathbb{L}_{N\times\mathbb{C}/A}$$

  relative to A.
- The morphism $\mathbb{L}_{N\times\mathbb{C}/A} \to \mathbb{L}_q$ of cotangent complexes induces $E \oplus \mathcal{O}_N \to \mathbb{L}_q$.
- Its cocone $\mathbb{E}$ induces a morphism $\mathbb{E} \to q^*\mathbb{L}_{\overline{N}/A}$.
- Using $D_{\mathbb{C}*}(N \times \mathbb{C} - M) \cong D(\overline{N})$, we obtain $\overline{E} := (q^*)^{-1}\mathbb{E} \to \mathbb{L}_{\overline{N}/A}$. We can check this is a perfect obstruction theory, extending E.

Torus localisation

- The fiberwise action on N and the trivial action on $\mathbb{C}$ induces a nontrivial action on $\overline{N}$.
- Since $\overline{N}$ is compact, the torus localised virtual class $[\overline{N}]^{vir}_{\mathsf{T}}$ pushes forward to $[\overline{N}]^{vir}$.
- Fixed loci is $M \cup \mathbb{P}(N)$. So the cycle $[\overline{N}]^{vir}_{\mathsf{T}}$ has contributions on M and $\mathbb{P}(N)$.
- The contribution on M is $[N]^{vir}_{\mathsf{T}}$.
- The contribution on $\mathbb{P}(N)$ is zero since the fixed part of $\overline{E}|_{\mathbb{P}(N)}$ has negative rank.
- So the pushforward of $[\overline{N}]^{vir}_{\mathsf{T}} = [N]^{vir}_{\mathsf{T}}$ into $\overline{N}$ and the pushdown to M is itself, on the other hand it is $\pi_*[\overline{N}]^{vir}$.

Extension $\overline{\sigma}$

- The key starting point to define σ was the tautological morphism $E_2 \to \mathcal{O}_N$.
- Recall that N is $h^0(E_2^\vee)$. So when we write $E_2^\vee = [F_0 \xrightarrow{d} F_1]$, N is cut-out space by $d \circ \tau_{F_0} : F_0 \to F_1|_{F_0}$, where τ_{F_0} is the tuatological section.
- Tautological section is extended to a section in the Euler sequence

$$0 \to \mathcal{O}_{\overline{F_0}}(-1) \xrightarrow{\overline{\tau} \oplus can} F_0 \oplus \mathcal{O}_{\overline{F_0}} \to T_q(-1) \to 0.$$

- Then $\overline{N}$ is cut-out by $d \circ \overline{\tau}$.
- As τ_{F_0} induced $E_2 \to \mathcal{O}_N$, $\overline{\tau}$ induces $\overline{E}_2 \to \mathcal{O}_{\overline{N}}(\mathbb{P}(N))$.

- We assume $E_1^\vee[1] \to E_2$ on N is extended to $\overline{N}$.
- For instance if $E_1^\vee[1] \to E_2$ is defined on M, it is obvious. This is the case of our examples.
- All these define $\overline{\sigma} : \overline{E}^\vee \to \mathcal{O}_{\overline{N}}(\mathbb{P}(N))[-1]$.
- $h^1(\overline{\sigma})$ is zero on M and has simple pole on $\mathbb{P}(N)$, where $[\overline{N}]^{vir}_{\overline{\sigma}}$ has nontrivial contributions.
- The contribution on M is $[N]^{vir}_{\sigma}$.

- Even if the fixed part of $\overline{E}|_{\mathbb{P}(N)}$ has negative rank, it defines a perfect obstruction theory of $\mathbb{P}(N)$. Note that its rank is -1.
- The composition with the restriction of $\overline{\sigma}$ is surjective in h^1, so we can define the reduced class $[\mathbb{P}(N)]^{red}$ of degree $2\text{rank} + 2 = 0$.
- We compute the contribution of $[\overline{N}]^{vir}_{\overline{\sigma}}$ on $\mathbb{P}(N)$ is $[\mathbb{P}(N)]^{red}$.
- It proves $[\overline{N}]^{vir}_{\overline{\sigma}} = [N]^{vir}_{\sigma} + [\mathbb{P}(N)]^{red}$.
- Pushing down to M is $\pi_*[\overline{N}]^{vir} = [N]^{vir}_{\sigma} + p_*[\mathbb{P}(N)]^{red}$

Thank you very much for your attention!