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Day-to-Day Flow Dynamics for Stochastic User Equilibrium and a General Lyapunov Function

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Abstract. This study establishes a general framework for continuous day-to-day models to capture the perceptual errors in travelers' day-to-day route choice behavior. As the counterpart of the Beckmann transformation, which has been widely used as a candidate Lyapunov function to prove the stability of continuous day-to-day traffic evolution models that converge to deterministic user equilibrium, Fisk's formulation is utilized in our study as a general Lyapunov function for the day-to-day models that converge to stochastic user equilibrium (SUE), so far as the path flow growth rates and the "potentials" of the paths satisfy the condition of negative correlation. A sufficient condition that guarantees the nonnegativity of the path flow is also provided. The logit dynamic, the logit-based Smith dynamic, and the logit-based Brown-von Neumann-Nash (BNN) dynamic are given as three examples under this framework. Moreover, we extend the second-order day-to-day model proposed by Xiao et al. [Xiao F, Yang H, Ye H (2016) Physics of day-to-day network flow dynamics. *Transportation Res. Part B: Methodological* 86:86–103.] for SUE. Some properties of the new model, such as fixed point and stability, are investigated. Interestingly, we find that even when the model converges to SUE, the path flows could still go negative during the oscillation under extreme situations. A numerical experiment is conducted to demonstrate the existence of negative path flow for the second-order model.

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Keywords: day-to-day • stochastic user equilibrium • Lyapunov function • stability

1. Introduction

The static traffic assignment models focus on the final state of distribution of origin–destination (OD) demand throughout the whole network. It is commonly known that the interaction (i.e., path choice behavior) between different paths as well as the road condition (i.e., link performance function) on the whole network together determine the final equilibrium states and corresponding travel times. It is reasonable to assume that a single traveler is always trying to minimize the travel cost with respect to his or her path choice. A stable status is reached when all the utilized paths between the same OD pair experience the same travel time that is no greater than any of the unused counterpart's. Such a status is characterized by the well-known deterministic user equilibrium (DUE) condition (Sheffi 1984). The DUE condition requires that the travelers possess the full information of the whole transportation network, and travelers are presumed to be able to make the right choice based on the full information. These strong presumptions can be relaxed by introducing the notion of "perceived travel cost," which can be viewed as a random variable related to the actual travel cost. From the random utility theory, travelers should choose the alternatives that yield the highest utility between OD pairs, and they always have inaccurate and distorted perception of path travel costs. By incorporating a random component in travelers' perception of travel cost, the choice interaction between different paths finally results in stochastic user equilibrium (SUE) (Daganzo and Sheffi 1977). There are two classical convex optimization problems solving DUE and SUE, respectively. One is called the Beckmann transformation (Beckmann et al. 1956), and the other is Fisk's formulation (Fisk 1980).

In a real transportation network, it is always observed that some events, such as traffic incidents, demand variation, capacity modifications, and network changes, would lead to the change of traffic flow pattern from equilibrium to disequilibrium (Kumar and Peeta 2015).

For such cases, the day-to-day models are required to explain the mechanism of network flow evolution and the possibility of approaching DUE or SUE state. Under a day-to-day setting, travelers make decisions today based on their knowledge of the previous evolution of the aggregate traffic flows, which itself depends on each individual's choice. For mathematical convenience, many day-to-day models that converge to DUE were established with continuous variables (Smith 1984, Nagurney and Zhang 1997, Cho and Hwang 2005, Han and Du 2012, He and Liu 2012, Guo et al. 2015). As the counterpart of continuous day-to-day models that converge to DUE, the continuous SUE models also attracted more and more attention recently in the day-to-day literature (Watling 1999, Smith and Watling 2016). The logit dynamic, which first appeared in Fudenberg and Levine (1998), describes players' aggregated strategies evolving toward the logit equilibrium. Different from this classic model, Smith and Watling (2016) extend the Smith dynamic to logit-based Smith dynamic to capture the perceptual errors in travelers' evaluations of travel cost. The fixed point is also SUE, and the stability analysis is conducted by using a quadratic Lyapunov function, similar to that adopted by Smith (1984). The difference between the two trajectories of the logit-based Smith dynamic and logit dynamic was shown by a simple numerical example.

Most ordinary differential equation–based continuous day-to-day models satisfy the Markov property. One can make the prediction of travelers' path choices in the next day solely on the basis of the previous day's traffic flow pattern under different hypothesized flow swapping rules; that is, travelers do not rely on the information that they achieved in the past (travelers are memoryless). However, such a simplification may not be very realistic because the historical traffic information one possesses would undoubtedly influence his/her current route choice. Compared with most continuous models that omit the influence of historical information, the role of past experience on travelers' path choice behavior is widely studied in the discrete model accompanied with a well-known learning filter (exponential smoothing filter). In a real transportation network, the discrete model is more desirable because the real-time system of repeated daily trips is indisputably discrete. Since the seminal paper by Horowitz (1984), the discrete evolution models have attracted great attention (Cascetta and Cantarella 1993, Cantarella and Cascetta 1995, Watling 1999, Bie and Lo 2010, Watling and Cantarella 2013). These perception-based models assume that travelers possess their own prior knowledge of the transportation network conditions and make path choices on the basis of their best knowledge. Travelers' knowledge of the network is normally represented by their perceived costs on individual paths, which could be updated according to the new trip experience and the real-time traffic information until the actual travel costs are identical to the perceived ones. The network flow loading process is executed using SUE assignment after the update of the whole network knowledge. In addition to describing the influence of past information on travelers' route choices, some literature also tried to model travelers' behavior in predicting others' route choices. On the basis of the notion of strategic thinking of rational players in behavior game theory, He and Peeta (2016) modeled drivers' prediction of others' response to the current traffic situation by formulating the marginal cost. He and Liu (2012) incorporated the commuters' prediction behavior by proposing a prediction-correction model and calibrated the proposed model by field data collected after the collapse of the I-35W bridge.

Looking at the literature of day-to-day models and evolutionary games, the Beckmann transformation has been endowed with some interesting physical meaning. In game theory, the traffic assignment models can be regarded as typical examples of "potential games." In a potential game, the incentives of all players' strategy benefits are mapped into one global potential function. From this point of view, Beckmann transformation can be assigned a physical-like meaning of "potential" for a transportation system. Beckmann transformation has also been used as a candidate Lyapunov function to prove the stabilities of quite a few day-to-day systems (Peeta and Yang 2003, Jin 2007, Kumar and Peeta 2015), so far as they follow the rational behavior adjustment process (RBAP), which states that "the aggregated travel cost in the system based on previous day's path travel costs will decrease when the path flow evolves from day to day" (Yang and Zhang 2009). Five typical continuous dynamical models—the proportional-switch adjustment (Smith 1984), the network tâtonnement process (Friesz et al. 1994), the projected dynamical system (Nagurney and Zhang 1997), the evolutionary traffic dynamic (Sandholm 2001), and the simplex gravity flow dynamic (Smith 1983)—are proved to follow RBAP. For these day-to-day models, the value of Beckmann transformation would always decrease at each time step as the time evolves and reaches its minimum at the equilibrium point, which can be analogous to the minimum total potential energy principle in physics. Very recently, Xiao et al. (2016) constructed a second-order flow-based day-to-day dynamic from the combination of travelers' learning process and flow swapping process. This study endowed an explicit physical meaning to this objective function by investigating the interaction between the defined kinetic energy and potential energy in a transportation network.

In this paper, by defining "potential" of a path in a transportation network, we establish a class of day-to-day models that converge to SUE. Under this framework, Fisk's formulation can be utilized as a general Lyapunov function for those dynamical systems. At SUE, the potential of every route is equal and the Lyapunov function is minimized. We show that the logit dynamic (Sandholm 2010) and the logit-based Smith dynamic (Smith and Watling 2016) both belong to this framework. The only difference between them is that the logit dynamic adopts "logit choice" based on potentials of the paths, whereas the logit-based Smith dynamic is based on pairwise comparison between the potentials of all routes. We also extend the second-order model in Xiao et al. (2016) by replacing the actual travel cost with potential to capture the randomness in the route-choice process. The fixed point of the new dynamical system then transfers from DUE to SUE. We show that a new Lyapunov function can be used to demonstrate the stability of the system. However, we demonstrate that the new system faces the similar problem of negative flow with the model in Xiao et al. (2016), even when all the path flows are positive at SUE.

The rest of this paper is organized as follows. Section 2 introduces the concept of stochastic traffic assignment. Section 3 establishes a general day-to-day dynamic model for SUE based on the definition of revision protocol and shows that the logit-based Smith dynamic and the logit dynamic are both special cases. Besides, a logit-based Brown–von Neumann–Nash (BNN) dynamic is developed based on BNN dynamic (Brown and Von Neumann 1950). Fisk's formulation is proved to be a general Lyapunov function for the above dynamical systems. The analogy of the system to the phase equilibrium in thermodynamics is also presented. Section 4 introduces a new second-order dynamical system that converges to SUE. The Lyapunov function is constructed by adding the entropy function to the total energy function defined in Xiao et al. (2016); and the existence of negative flow is proved. The last section concludes the whole study and highlights some future directions.

2. The Stochastic Traffic Assignment Problem

To begin with, we first introduce the traditional stochastic traffic assignment problem with separable link cost functions. Suppose a directed traffic network consists of W OD pairs with positive traffic demands d_w, where N donates the set of nodes and A the set of links. Let R_w denote the set of all paths connecting OD pair $w \in W$ and $|R_w|$ denote the number of elements in R_w. Furthermore, the flow and cost on path $r \in R_w$ between OD pair $w \in W$ are denoted by f_{rw} and c_{rw}, and the flow and cost on link $a \in A$ are denoted by v_a and c_a. Let the matrices $\Delta = [\delta_{a,rw}]$ and $\Lambda = [\lambda_{rw}]$ denote the link-path and OD-path incidence matrices, respectively, where $\delta_{a,rw}$ equals 1 if path r uses link a and 0 otherwise, and λ_{rw} equals 1 if path r connects OD pair w and 0 otherwise. Then the path cost c_{rw} can be expressed by $c_{rw} = \sum_{a \in A} c_a \delta_{a,rw}$. The feasible solution set of path flows can then be expressed by

$$\Omega \triangleq \{\mathbf{f} | \ \mathbf{v} = \Delta \mathbf{f}, \mathbf{d} = \Lambda \mathbf{f}, \mathbf{f} > \mathbf{0}\}. \tag{1}$$

Following Fisk (1980), we can obtain the fixed-demand SUE by the following minimization problem,

$$G(\mathbf{f}) = \min \sum_{a \in A} \int_0^{v_a} c_a(\omega) d\omega + \sum_{w \in W} \sum_{r \in R_w} \theta_w f_{rw} \ln(f_{rw}) \tag{2}$$

subject to Ω, where θ_w is a dispersal parameter called the noise level. As shown by Fisk (1980), it can be derived from the Kuhn–Tucker conditions of the optimization problem that

$$\frac{f_{sw}}{f_{rw}} = \frac{\exp(-c_{sw}/\theta_w)}{\exp(-c_{rw}/\theta_w)}, \quad \forall r, s \in R_w, w \in W \tag{3}$$

which is exactly SUE. If cost function c is continuous and monotone on Ω, (2) is strictly convex, so that there exists a unique SUE solution (Cantarella and Cascetta 1995).

3. A Framework for SUE Day-to-Day Dynamics Without User Learning

Based on the revision protocols, a general form of day-to-day dynamic for SUE is developed in this section. We illustrate how the well-known logit dynamic (Fudenberg and Levine 1998) and the logit-based Smith dynamic (Smith and Watling 2016) can be included in this framework. Moreover, we build an additional logit-based BNN dynamic within this framework, which extends the BNN dynamic (Brown and Von Neumann 1950) from DUE to SUE. A general Lyapunov function (Fisk's formulation) is proposed to prove the asymptotic stability of the system.

3.1. A General Day-to-Day Model for SUE

We first define the "potential" of a path $r \in R_w$, μ_{rw}, by taking the partial derivative of (2) with respect to path flow f_{rw},

$$\mu_{rw} = c_{rw} + \theta_w(\ln f_{rw} + 1). \tag{4}$$

Because the noise level θ_w is a parameter only dependent on OD pair w, without loss of generality, the constant "1" can be ignored in (4). Then (4) can be simplified as

$$\mu_{rw} = c_{rw} + \theta_w \ln f_{rw}. \tag{5}$$

With the above definitions, we have the following theorem for the relationship between path potential and SUE.

Lemma 1. *At SUE, the potentials of all the paths connecting the same OD pair are equal.*

Proof. The proof is straightforward. Substituting (3) into (5), we get

$$\mu_{rw} = \mu_{sw} \qquad \forall r, s \in R_w, w \in W. \tag{6}$$

That is, the potentials are equal for all the paths connecting the same OD at SUE. □

To capture drivers' perception error when choosing paths, we redefine the revision protocols in Sandholm (2010) with the noise level, $\boldsymbol{\theta}$.

Definition 1. Let $\rho(\mathbf{c}, \boldsymbol{\theta}, \mathbf{f}) \geq 0$ be a revision protocol that depends on path cost, $\mathbf{c}$, path flow, $\mathbf{f}$, and noise level, $\boldsymbol{\theta}$, where $\rho_{rs,w}(\mathbf{c}_w, \boldsymbol{\theta}_w, \mathbf{f}_w)$ describes the swapping rate of each traveler from path r to path s connecting OD pair w.

Then the mean dynamic corresponding to revision protocol, ρ, can be built as below

$$\frac{df_{rw}}{dt} = \sum_{s \in R_w} f_{sw}\rho_{sr,w} - f_{rw}\sum_{s \in R_w}\rho_{rs,w}, \quad \forall r \in R_w, w \in W. \tag{7}$$

The microscopic interpretation is as follows: assume every driver is equipped with a stochastic alarm clock and the times between rings of drivers' clocks are independent. The ringing of a clock signals the arrival of a revision opportunity for the clock's owner. The owner will make their decision corresponding to revision protocol. For more details, readers may refer to Sandholm (2010). Next, we will show three examples that are special cases of the mean dynamic (7).

3.1.1. The Logit-Based Smith Dynamic. Suppose the revision protocol ρ has the following form

$$\rho_{sr,w} = \alpha_w \max(0, \mu_{sw} - \mu_{rw}), \qquad \forall r, s \in R_w, w \in W, \tag{8}$$

where $\alpha_w > 0$ is a scalar coefficient, which is assumed to be only dependent on OD pair w. Equation (8) implies that there is only one-way route swapping from the route with high potential to the route with low potential; and the swapping rate is proportional to the difference between the two potentials of the routes. Substituting (5) and (8) into (7), we get the logit-based Smith dynamic,

$$\frac{df_{rw}}{dt} = \alpha_w\left(\sum_{s \in R_w} f_{sw}\max(0, c_{sw} + \theta_w \ln f_{sw} - c_{rw} - \theta_w \ln f_{rw}) - \sum_{s \in R_w} f_{rw}\max(0, c_{rw} + \theta_w \ln f_{rw} - c_{sw} - \theta_w \ln f_{sw})\right). \tag{9}$$

3.1.2. The Logit Dynamic. Assume the revision protocol ρ does not depend on drivers' current strategies; in other words, the probabilities that travelers on different paths switch to the same path are equal. Then, $\rho_{sr,w}$ can be replaced by $\alpha_w \tau_{rw}$, where $\rho_{sr,w} := \alpha_w \tau_{rw}$. From Equation (7), we have

$$\frac{df_{rw}}{dt} = \alpha_w\left(d_w\tau_{rw} - f_{rw}\sum_{s \in R_w}\tau_{sw}\right), \forall r, s \in R_w, w \in W. \tag{10}$$

We further assume that $\sum_{s \in R_w}\tau_{sw} = 1, \forall r, s \in R_w, w \in W$. This assumption is not very restrictive, beause the only effect of replacing τ_{sw} with its scalar multiple $\tau_{rw}/\sum_{s \in R_w}\tau_{sw}$ is to change the speed at which the evolutionary process runs by a constant factor and the scalar multiple can be incorporated into α_w. Such dynamic is called exact target dynamic (Sandholm 2010). When $\sum_{s \in R_w}\tau_{sw} = 1$, Equation (10) can be written as

$$\frac{df_{rw}}{dt} = \alpha_w(d_w\tau_{rw} - f_{rw}), \forall r, s \in R_w, w \in W. \tag{11}$$

Here τ_{rw} can be regarded as a probability that the travelers will choose path r. From the perspective of random utility theory, travelers may have inaccurate and distorted perceptions of path travel costs in which the "perceived travel cost" is constructed by adding stochastic perturbations to the actual travel cost of each path. At time $t + \delta t$, a traveler chooses the best response to the vector of travel cost, $c(t)$, that has been perturbed by a random vector $\varepsilon(t)$. Then the probability that path $r \in R_w$ would be chosen can be defined as

$$\tau_{rw}(\mathbf{f}) = P\left(r = \underset{s \in R_w}{\operatorname{argmin}}(c_{sw}(\mathbf{f}) + \varepsilon_s)\right). \tag{12}$$

If the random terms ε_s are independently and identically distributed Gumbel variates, τ_{rw} can be calculated by

$$\tau_{rw}(\mathbf{f}) = \frac{\exp(-c_{rw}/\theta_w)}{\sum_{s \in R_w}\exp(-c_{sw}/\theta_w)}. \tag{13}$$

Substituting (13) into (11) yields the logit dynamic

$$\frac{df_{rw}}{dt} = \alpha_w\left(d_w\frac{\exp(-c_{rw}/\theta_w)}{\sum_{s \in R_w}\exp(-c_{sw}/\theta_w)} - f_{rw}\right). \tag{14}$$

To understand the logit dynamic from the perspective of path potential, μ_{rw}, let $\mathbf{f}^*_w = d_w\boldsymbol{\tau}_w$, and $\mathbf{f}^*_w$ is determined by the following problem

$$\mathbf{f}^*_w = \underset{\mathbf{f} \in \Omega_w}{\operatorname{argmin}} \sum_{r \in R_w} f_{rw}\mu_{rw}, \tag{15}$$

where $\mu_{rw}(f_{rw}) = c_{rw} + \theta_w \ln f_{rw}$, is the potential of path r, except that here c_{rw} is fixed and not dependent on f_{rw}. We then have

$$\frac{df_{rw}}{dt} = \alpha_w(f^*_{rw} - f_{rw}). \tag{16}$$

The dynamical system (16) indicates that the path flow pattern would move from the current flow pattern $\mathbf{f}$ toward a target flow $\mathbf{f}^*$ pattern at each day-to-day time, and $\mathbf{f}^*$ is determined by (15).

It is worth noting that here Equation (15) is equivalent to finding the SUE at each day-to-day time with fixed link travel cost. The dynamical system (15)–(16) formulates the flow dynamic by defining the target flow pattern that is determined by both the travelers' cost-minimization behaviors and reluctance to change routes. The cost-minimization behavior is essentially to find the path with lowest potential for each OD pair, and the inertia effect existing in travelers is captured by defining the distance between the target flow pattern and current flow pattern.

Lemma 2. *The dynamical system* (15)–(16) *coincides with the logit dynamic model.*

Proof. The Lagrangian of the minimization problem (15) can be written as

$$L(\mathbf{f}^*,\gamma)=\sum_{r\in R_w} f^*_{rw}c_{rw}+\sum_{w\in W}\theta_w\sum_{r\in R_w} f^*_{rw}\ln f^*_{rw} + \sum_{w\in W}\gamma_w\Big(d_w-\sum_{r\in R_w} f^*_{rw}\Big). \tag{17}$$

The first-order conditions are

$$f^*_{rw}\frac{\partial L(\mathbf{f}^*,\gamma)}{\partial f^*_{rw}} = f^*_{rw}(c_{rw}+\theta_w\ln f^*_{rw}+\theta_w-\gamma_w)=0, \tag{18}$$

$$f^*_{rw}>0, c_{rw}+\theta_w\ln f^*_{rw}+\theta_w-\gamma_w=0, \tag{19}$$

$$\frac{\partial L(\mathbf{f}^*,\gamma)}{\partial \gamma_w}=d_w-\sum_{r\in R_w} f^*_{rw}=0. \tag{20}$$

With (18)–(20), we have

$$f^*_{rw}=d_w\frac{\exp(-\theta_w c_{rw})}{\sum_{s\in R_w}\exp(-\theta_w c_{sw})}. \tag{21}$$

Substituting (21) into (16) we have

$$\frac{df_{rw}}{dt}=\alpha_w\left(d_w\frac{\exp(-\theta_w c_{rw})}{\sum_{s\in R_w}\exp(-\theta_w c_{sw})}-f_{rw}\right), \tag{22}$$

which is exactly the logit dynamic. □

3.1.3. The Logit-Based BNN Dynamic. Without considering the perceived travel cost, Yang (2005) assumed that the swapping rate is proportional to the sum of excess path cost

$$\tau_{rw}(\mathbf{f})=\max\{0,\ -c_{rw}+\bar{c}_w\}, \tag{23}$$

where $\bar{c}_w=\left(\sum_{r\in R_w} c_{rw} f_{rw}\right)/d_w$. Substituting (23) into (7), we get the BNN dynamic

$$\frac{df_{rw}}{dt}=\alpha_w\left(d_w\max\{0,\bar{c}_w-c_{rw}\}-f_{rw}\sum_{s\in R_w}\max\{0,\bar{c}_w-c_{sw}\}\right). \tag{24}$$

To extend the BNN dynamic from DUE into SUE with consideration of the perceived travel cost, a reasonable assumption is that the swapping rate is proportional to the sum of excess potential

$$\tau_{rw}(\mathbf{f})=\max\{0,\ -\mu_{rw}+\bar{\mu}_w\}, \tag{25}$$

where $\bar{\mu}_w=\left(\sum_{r\in R_w}\mu_{rw} f_{rw}\right)/d_w$. Substituting (5) and (25) into (7) yields the logit-based BNN dynamic

$$\frac{df_{rw}}{dt}=\alpha_w\left(d_w\max\{0,\bar{\mu}_w-\mu_{rw}\}-f_{rw}\sum_{s\in R_w}\max\{0,\bar{\mu}_w-\mu_{sw}\}\right). \tag{26}$$

3.2. A General Lyapunov Function

Beckmann et al. (1956) constructed a mathematical minimization problem to calculate the static user equilibrium, which is now called "Beckmann transformation"

$$\min\sum_{a\in A}\int_0^{v_a} c_a(\omega)d\omega. \tag{27}$$

In the day-to-day literature, many studies (Peeta and Yang 2003, Jin 2007, Kumar and Peeta 2015) have proved the asymptotic stability of their day-to-day models by viewing Beckmann transformation as the Lyapunov candidate function. In this section, we will prove that, similar to the day-to-day models converging to DUE, Fisk's formulation can be utilized as a general candidate Lyapunov function for the day-to-day model (7) for SUE, which includes the logit dynamic, the logit-based Smith dynamic, and the logit-based BNN dynamic.

Proposition 1. *If the fixed point of a day-to-day dynamic coincides with SUE and the following condition holds*

$$\dot{\mathbf{f}}(t)\begin{cases}\in\Phi \text{ if } & \Phi\neq\varnothing\\ =0 \text{ if } & \Phi=\varnothing\end{cases} \text{ where } \Phi=\{\dot{\mathbf{f}}(t):\boldsymbol{\mu}^T\dot{\mathbf{f}}(t)<0\},\ \forall t>t_0, \tag{28}$$

then the function

$$V_1(\mathbf{f})=G-G_{min} \tag{29}$$

can be a candidate Lyapunov function and the day-to-day dynamic asymptotically converges to SUE.

Proof. Substituting (2) into (29) and taking the first-order derivative, we have

$$\frac{dV_1(\mathbf{f})}{dt}=\frac{\partial}{\partial t}\left(\sum_{a\in A}\int_0^{v_a(\mathbf{f})}c_a(\omega)\mathrm{d}\omega\right) + \sum_{w\in W}\sum_{r\in R_w}\frac{\partial}{\partial t}(\theta_w f_{rw}\ln f_{rw}) \tag{30}$$

$$=\sum_{w\in W}\sum_{r\in R_w}c_{rw}\dot{f}_{rw}+\sum_{w\in W}\sum_{r\in R_w}\theta_w(\ln f_{rw}+1)\dot{f}_{rw} \tag{31}$$

$$=\sum_{w\in W}\sum_{r\in R_w}\mu_{rw}\dot{f}_{rw}+\sum_{w\in W}\theta_w\sum_{r\in R_w}\dot{f}_{rw}. \tag{32}$$

Here (31) requires that the link cost functions are separable (i.e., the travel cost of a link is only dependent on its own flow). From the flow conservation condition, we know that $\sum_{r\in R_w} \dot{f}_{rw} = 0$, *for* $\forall w \in W$. Thus, we have

$$\frac{dV_1(\mathbf{f})}{dt} = \sum_{w\in W} (\boldsymbol{\mu}_w)^T \dot{\mathbf{f}}_w(t) \leq 0. \tag{33}$$

Equations (33) can be obtained by condition (28), and the equality holds if and only if the day-to-day dynamic reaches SUE. Function G monotonically decreases, and we have

$$V_1(\mathbf{f}) \geq 0, \tag{34}$$

where the equality holds if and only if the dynamic reaches SUE. Then by Lyapunov's second theorem (Sell 1962), the day-to-day dynamic model asymptotically converges to SUE. □

Lemma 3. *The logit-based Smith dynamic satisfies the condition* (28).

Proof. The fixed point of the logit-based Smith dynamic coincides with SUE, and this has been proved in Smith and Watling (2016). Here we show that the logit-based Smith dynamic meets the condition (28). For each OD pair w, we number the paths according to their potentials in a decreasing order, then we have $\mu_{1w} \geq \mu_{2w} \geq \cdots \geq \mu_{|R_w|w}$. Then we have

$$\begin{aligned}
\sum_{i=1}^{|R_w|} \mu_{iw} \dot{f}_{iw} &= \mu_{1w}\Big(-\sum_{i=2}^{|R_w|} \alpha_w f_{1w}(\mu_{1w} - \mu_{iw})\Big) \\
&\quad + \mu_{2w}\Big(-\sum_{i=3}^{|R_w|} \alpha_w f_{2w}(\mu_{2w} - \mu_{iw}) \\
&\quad + \sum_{i=1}^{1} \alpha_w f_{iw}(\mu_{iw} - \mu_{2w})\Big) + \cdots \\
&\quad + \mu_{|R_w|w} \sum_{i=1}^{|R_w|-1} \alpha_w f_{iw}(\mu_{iw} - \mu_{|R_w|w}) \\
&= -\sum_{i=2}^{|R_w|} \alpha_w f_{1w}(\mu_{1w} - \mu_{iw})^2 \\
&\quad - \sum_{i=3}^{|R_w|} \alpha_w f_{r_2 w}(\mu_{2w} - \mu_{iw})^2 - \cdots \\
&\quad - \sum_{i=|R_w|}^{|R_w|} \alpha_w f_{(|R_w|-1)w}(\mu_{(|R_w|-1)w} - \mu_{iw})^2 \leq 0.
\end{aligned} \tag{35}$$

Because (35) holds for any $w \in W$, we have $\boldsymbol{\mu}^T \dot{\mathbf{f}}(t) \leq 0$, and the equality holds if and only if $\mu_{1w} = \mu_{2w} = \cdots = \mu_{|R_w|w}$, which represents SUE. □

Lemma 4. *The logit dynamic satisfies the condition* (28).

Proof. See theorem 6.2.10 in Sandholm (2010) with the notion of virtual positive correlation. □

Lemma 5. *The logit-based BNN dynamic satisfies the condition* (28).

Proof. For $\forall w \in W$, we can define

$$\Psi_w = \{\forall r \in w, \bar{\mu}_w > \mu_{rw}\}. \tag{36}$$

Then

$$\tau_{rw} = \begin{cases} -\mu_{rw} + \bar{\mu}_w & \text{if} \quad r \in \Psi_w \\ 0 & \text{if} \quad r \notin \Psi_w \end{cases}. \tag{37}$$

From the logit-based BNN dynamic (26), we have

$$\begin{aligned}
\sum_{i=1}^{|R_w|} \mu_{iw} \dot{f}_{iw} &= \sum_{r\in R_w} \Big(\mu_{rw}\Big(d_w \tau_{rw} - f_{rw} \sum_{s\in R_w} \tau_{sw}\Big)\Big) \\
&= d_w \sum_{r\in R_w} (\mu_{rw}\tau_{rw}) - \Big(\sum_{s\in R_w} \tau_{sw}\Big) \sum_{r\in R_w} (\mu_{rw} f_{rw}) \\
&= d_w \sum_{r\in R_w} (\mu_{rw}\tau_{rw}) - \Big(\sum_{s\in R_w} \tau_{sw}\Big)(\bar{\mu}_w d_w) \\
&= d_w \Big(\sum_{r\in R_w} (\mu_{rw}\tau_{rw}) - \bar{\mu}_w \Big(\sum_{s\in R_w} \tau_{sw}\Big)\Big) \\
&= d_w \Big(\sum_{r\in \Psi_w} (\mu_{rw}(-\mu_{rw} + \bar{\mu}_w)) \\
&\qquad - \bar{\mu}_w \Big(\sum_{r\in \Psi_w} (-\mu_{rw} + \bar{\mu}_w)\Big)\Big) \\
&= d_w \Big(\sum_{r\in \Psi_w} -(\mu_{rw} - \bar{\mu}_w)^2\Big) \leq 0.
\end{aligned} \tag{38}$$

Because (38) holds for any $w \in W$, we have $\boldsymbol{\mu}^T \dot{\mathbf{f}}(t) \leq 0$, and the equality holds if and only if $\mu_{1w} = \mu_{2w} = \cdots = \mu_{|R_w|w}$, which represents SUE. □

From Lemmas 3–5 and Proposition 1, it is straightforward to conclude that $V_1(\mathbf{f})$ is a candidate Lyapunov function for the logit-based Smith dynamic, the logit dynamic, and the logit-based BNN dynamic. As a result, the three dynamics asymptotically converge to SUE.

Lemma 6. *Let* $f_{r_{min}w}, r_{min} \in R_w, w \in W$ *be the minimum path flow in the network at any time* $t \geq t_0$. *If* $\lim_{f_{r_{min}w} \to 0^+} f_{r_{min}w} \cdot \sum_{s\in R_w} \rho_{r_{min}s,w} = 0$, *for* $\forall t > t_0$, *then under the day-to-day dynamic* (7), $\mathbf{f} > 0$.

Proof. From definition, $\rho \geq 0$. Then at any time t, if $\mathbf{f}(t) > 0$, we have $\sum_{s\in R_w} f_{sw}\, \rho_{sr_{min},w} \geq 0$; and from the assumption we also have $\lim_{f_{r_{min}w} \to 0^+} f_{r_{min}w} \sum_{s\in R_w} \rho_{r_{min}s,w} = 0$. Combining the two and the day-to-day dynamic (7), we have

$$\lim_{f_{r_{min}w} \to 0^+} \frac{df_{r_{min}w}}{dt} = \lim_{f_{r_{min}w} \to 0^+} \sum_{s\in R_w} f_{sw} \rho_{sr_{min},w} \geq 0, \quad \forall r \in R_w. \tag{39}$$

Because (39) holds for any time t, we have that $\mathbf{f}(t) > 0$, *for* $\forall t > t_0$. □

It is not hard to prove that the logit-based Smith dynamic, the logit dynamic, and the logit-based BNN dynamic all follow the assumption of Lemma 6 (Online Appendix A). Thus, the path flows can never turn negative for the three models during the flow oscillation.

3.3. Analogy to Thermodynamics

In the literature of traffic assignment models, initially, Beckmann's transformation was not endowed any intuitive economic or behavioral interpretation and was only viewed strictly as a mathematical construct to solve DUE (Sheffi 1984). However, in the day-to-day literature, some studies (Peeta and Yang 2003, Jin 2007, Kumar and Peeta 2015, Xiao et al. 2016) have proved the asymptotic stability of their day-to-day models by viewing Beckmann transformation as the Lyapunov candidate function, which assigned physical meaning to this objective function: the potential energy. The potential energy of the transportation network would always decrease at each time step as the time evolves and reaches its minimum at the equilibrium point, which can be generalized by *the minimum total potential energy principle* in physics. Inspired by the analogy between day-to-day dynamics for DUE and physical systems, in this section we reveal a similar analogy between the day-to-day dynamics for SUE and the chemical reaction in a mixture in thermodynamics. We show that Fisk's formulation can be interpreted as the synonymous "Gibbs free energy" of a transportation network. Similar to the RBAP defined in Yang and Zhang (2009) with respect to the deterministic day-to-day model, the Gibbs free energy decreases during the flow evolution. The equilibrium is obtained when the "chemical potential" of each route is equal and the Gibbs free energy reaches minimum.

To begin with, we introduce briefly the mathematical model of multiphase equilibrium. Suppose an equilibrated ice-water mixture in environment (T_1, P_1) is suddenly moved to a new environment (T_2, P_2), where T and P denote temperature and pressure of the mixture, respectively. Because of the temperature and pressure changes, the original equilibrium is broken. Then the composition proportions of ice and water would change, and the chemical reaction between them can be expressed as below

$$H_2O(l) \leftrightharpoons H_2O(s). \tag{40}$$

Chemical reaction (40) is reversible. Chemical equilibrium is reached when the reaction rates in both directions are equal.

Let $G_{ice-water}$ be the Gibbs free energy of the ice-water mixture, which is defined as

$$G_{ice-water} = X_l G_l + X_s G_s + K(X_l \ln X_l + X_s \ln X_s), \tag{41}$$

where X_l, X_s, denote the amounts of the two substances. G_l, G_s denote standard molar Gibbs free energy, and K is a constant at certain temperature and pressure. It is worth mentioning that G_l, G_s do not depend on X_l, X_s. So Equation (41) can also be written as

$$G_{ice-water} = \int_0^{X_l} G_l(\omega)d\omega + \int_0^{X_s} G_s(\omega)d\omega + K(X_l \ln X_l + X_s \ln X_s). \tag{42}$$

Chemical potential is known as partial molar free energy and is a form of potential energy that can be absorbed or released during a chemical reaction. From Equation (41) we can obtain the chemical potential of water molecule in the liquid phase, μ_l^{water}, and in the solid phase, μ_s^{water}, by taking the partial derivatives of $G_{ice-water}$ with respect to X_l and X_s, respectively.

$$\mu_l^{water} = G_l + K(\ln X_l + 1) \tag{43}$$

$$\mu_s^{water} = G_s + K(\ln X_s + 1). \tag{44}$$

According to the aggregate theory, water molecules tend to move from high chemical potential status to low chemical potential status, and the chemical equilibrium is reached when $\mu_l^{water} = \mu_s^{water}$. Because chemical potential is relative, without loss of generality, the constant "1" can be ignored in (43) and (44), and the chemical potential is usually defined as

$$\mu = G + K \ln X. \tag{45}$$

And we introduce the well-known *Gibbs free energy minimum principle*, which states that the Gibbs free energy will decrease and approach a minimum value at equilibrium in a closed system.

In a traffic network, each driver has his/her own perception of the current network state and makes route choices on the basis of the perceived path costs. It is interesting to see that the dynamic transportation system is similar to the multiphase system in the following ways:

1. They both have multiple "phases." In a traffic network, the multiple paths connecting the same OD pair act as the role of "phase."
2. Both processes are "reversible" and "spontaneous." In the transportation system, some drivers would spontaneously swap from path r to path s, and at the same time some drivers would spontaneously swap from path s to path r. The reason is that perceived travel cost is a random variable.

Inspired by the multiphase system in Thermodynamics, we will define the Gibbs free energy and chemical potential of a transportation network and investigate the flow dynamics and equilibrium from the perspective of energy.

We regard the path flow f_{rw} as the amount of substance and the path cost c_{rw} as the standard molar Gibbs free energy. By analogy to Equation (42), the "Gibbs free energy" of a day-to-day dynamical system, G_{tra}, can be defined as

$$G_{tra}(\mathbf{f}) = \sum_{a\in A}\int_0^{v_a} c_a(\omega)d\omega + \sum_{w\in W}\sum_{r\in R_w}\theta_w f_{rw}\ln(f_{rw}), \quad (46)$$

which is exactly taking the form of the "Fisk's formulation" in Equation (2). Similar to Equations (43) and (44), the chemical potential of a route, μ_{rw}, can be found by taking the partial derivative of (46) with respect to path flow f_{rw},

$$\mu_{rw} = c_{rw} + \theta_w(\ln f_{rw} + 1). \quad (47)$$

Definition (46) can be explained by analogy to the multiphase system in thermodynamics. In a mixture, the chemical potential of a component can be derived by taking the partial derivative of the total Gibbs energy of the system with respect to the amount of the component. In a transportation system, it should follow the similar definition, that is, the chemical potential of a route should be the partial derivative of the total Gibbs energy of the transportation system with respect to the traffic flow of the route. From this definition, Fisk's formulation becomes a "must" to define the total Gibbs energy of a transportation system (considering a general network topology). Similar to the multiphase system in thermodynamics, θ_w is a parameter only dependent on OD pair w. Without loss of generality, the constant "1" can be ignored in (47), and the chemical potential of a path can be simply defined as

$$\mu_{rw} = c_{rw} + \theta_w \ln f_{rw}, \quad (48)$$

which is consistent with the definition (5) we made before. Additionally, from Proposition 1, we know that the system will lose its "Gibbs free energy" gradually until it stops at SUE point, which is consistent with the *minimum Gibbs free energy principle* in thermodynamics.

4. A Second-Order Model for Day-to-Day Dynamics with Stochastic Route Choice and User Learning

In this section, we propose a new day-to-day model that extends the second-order model under DUE (Xiao et al. 2016) to the case of dynamical system converging to SUE, to capture the drivers' perceptual errors in evaluation of travel cost.

4.1. Formulation Second-Order Day-to-Day Model for SUE

Xiao et al. (2016) constructed a second-order flow-based day-to-day dynamic from the combination of travelers' learning process and flow switching process. The aggregate flow switching process is determined by pairwise comparison of perceived travel cost on each route. By analogizing to the damped harmonic oscillator system, the model showed that the dynamical system will eventually approach the DUE path flow pattern with the defined kinetic and potential energies decreasing to their minima. In this paper, we call this model the second-order DUE dynamical model, which is shown below

$$\ddot{f}_{rw} + \beta_w \dot{f}_{rw} - \alpha_w\beta_w \sum_{s\in R_w}(c_{sw} - c_{rw}) = 0, \quad (49)$$

where α_w represents travelers' sensitivity to the difference of travel costs between each two routes. β_w is the memory decay rate associated with OD pair w. The total energy of the system (49) is defined as the summation of potential energy and kinetic energy,

$$E(\mathbf{f}) = E_p + E_k = \sum_{a\in A}\int_0^{v_a(\mathbf{f})} c_a(\omega)d\omega + \sum_{w\in W}\sum_{r\in R_w}\frac{1}{2}m_w \dot{f}_{rw}^{\,2}, \quad (50)$$

where $m_w = 1/(\alpha_w\beta_w|R_w|)$ represents "mass" of each route. For more details, the reader can refer to Xiao et al. (2016). This model combines the learning process and flow swapping process from a natural perspective. However, it ignores the drivers' perceptual errors in evaluating travel cost. The purpose of this section is to extend the above model to the case of a dynamical system converging to SUE, to capture the randomness in the route-choice process. The pairwise comparison implied in this model is similar to the pairwise route swapping dynamic (Smith and Watling 2016), and by the analysis in Section 3.1 that the drivers tend to swap from the path with high potential to the path with low potential, the following second-order model for SUE is then constructed

$$\ddot{f}_{rw} + \beta_w \dot{f}_{rw} - \alpha_w\beta_w \sum_{s\in R_w}(\mu_{sw} - \mu_{rw}) = 0. \quad (51)$$

For high-order dynamical systems, it is convenient to degrade the order by letting $x_{1,rw} = f_{rw}$, $x_{2,rw} = \dot{x}_{1,rw}$. Then the flow evolution process (51) can be characterized by Equations (52)–(53)

$$\begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\end{bmatrix} = \begin{bmatrix}\mathbf{x}_2\\ \boldsymbol{\alpha\beta}\mathbf{g}(\mathbf{x}_1) - \boldsymbol{\beta}\mathbf{x}_2\end{bmatrix} \quad (52)$$

$$\mathbf{g}(\mathbf{x}_1) = (g_{rw}, r\in R_w, w\in R_w), \text{ where } g_{rw} = \sum_{s\in R_w}(\mu_{sw} - \mu_{rw}) \quad (53)$$

$\boldsymbol{\alpha}$ is a diagonal matrix with the element equals to α_w at (r, w) and 0 otherwise; $\boldsymbol{\beta}$ is a diagonal matrix with the element equals to β_w at (r, w) and 0 otherwise; $\boldsymbol{\theta}$ is a diagonal matrix with the element equal to θ_w at (r, w) and 0 otherwise. Here θ_w is defined by (5).

4.2. Existence of Negative Path Flow

To avoid negative path flow, assumption 1 in Xiao et al. (2016) assumes that the path flows only evolve in the interior of the feasible path flow set. Because at SUE every path will have positive flow, can this assumption be relaxed for dynamical system (52)–(53)? Unfortunately, in the following we will prove that even under the second-order dynamical system converging to SUE, the path flow could still penetrate 0 and turn negative under extreme situations.

Let us first define the convex set of demand-route flows

$$D = \{\mathbf{x_1}: \mathbf{x_1} \in R^n \text{ with } \mathbf{A}\mathbf{x_1} = \mathbf{q}\}, \tag{54}$$

where $\mathbf{A}$ denotes the OD-route incidence matrix. $\mathbf{q}$ is the OD demand vector. Then we define the set of n-vectors with all the path flows are positive

$$S = \{\mathbf{x_1}: \mathbf{x_1} \in R^n \text{ with } x_{1,rw} > 0 \text{ for } r = 1, 2, \ldots, n\}. \tag{55}$$

As we have proved before, the Lyapunov function will decrease as the dynamical system evolves. That is

$$V(\mathbf{x}(t)) \leq V(\mathbf{x}(0)). \tag{56}$$

Let $L = V(\mathbf{x}(0))$, then $V(\mathbf{x}(t)) \leq L$ for all $t \geq 0$, that is

$$\sum_{a\in A} \int_0^{v_a(\mathbf{x}_1(t))} c_a(\omega)d\omega + \sum_{r\in R_w} \theta_w(x_{1,rw}(t)\ln(x_{1,rw}(t))) + \sum_{w\in W}\sum_{r\in R_w} \frac{1}{2} m_w x_{2,rw}^2(t) \leq L. \tag{57}$$

The demand is fixed and cannot be arbitrarily large. Then the path flow is bounded by the OD demand. Moreover, $c_a(\cdot)$ is nonnegative, so the term $\sum_{a\in A}\int_0^{v_a(x)} c_a(\omega)d\omega$ is also bounded. Let us say

$$0 < \sum_{a\in A} \int_0^{v_a(\mathbf{x}_1(t))} c_a(\omega)d\omega \leq M. \tag{58}$$

When $x_{1,rw} \to 0$, $x_{1,rw}\ln(x_{1,rw}) \to 0$, and $x_{1,rw}\ln(x_{1,rw})$ reaches the minimum at $x_{1,rw} = 1/e$. As a result, when the demand is fixed, $\sum_{r\in R_w}\theta_w(x_{1,rw}\ln(x_{1,rw}))$ is also bounded. Let us say

$$-N_1 < \sum_{r\in R_w} \theta_w(x_{1,rw}(t)\ln(x_{1,rw}(t))) \leq N_2. \tag{59}$$

From Equations (57)–(59), we have

$$0 \leq \sum_{w\in W}\sum_{r\in R_w} \frac{1}{2} m_w x_{2,rw}^2(t) \leq L + N_1. \tag{60}$$

Then we have that the path flow change rate $x_{2,rw} = \dot{x}_{1,rw}$ at any time t is bounded

$$-B \leq x_{2,rw}(t) \leq B. \tag{61}$$

The dynamical system (52)–(53) states that for path r connecting OD pair w, the first-order derivative of path flow change rate is

$$\dot{x}_{2,rw}(t) = \alpha_w\beta_w \sum_{i\in R_w}(c_{iw}(t) - c_{rw}(t)) + \alpha_w\beta_w\theta_w \sum_{i\in R_w}(\ln x_{1,iw}(t) - \ln x_{1,rw}(t)) - \beta_w x_{2,rw}(t). \tag{62}$$

Let us look at the path with the smallest flow at any time t, $x_{1,min}(t)$, say it is between OD pair w; and we can also find the maximum flow $x_{1,max}(t)$ between OD pair w. Because the demand for OD pair w is bounded by d_w, we have $x_{1,max}(t) \leq d_w$. We assume that

$$-C \leq \alpha_w\beta_w \sum_{i\in R_w}(c_{iw}(t) - c_{rw}(t)) \leq C. \tag{63}$$

With (61)–(63), we have

$$\begin{aligned}\dot{x}_{2,min}(t) &\leq C + \alpha_w\beta_w\theta_w \sum_{i\in R_w}(\ln d_w - \ln x_{1,min}(t)) + \beta_w B \\ &= C + |R_w|\alpha_w\beta_w\theta_w \ln d_w + \beta_w B - |R_w|\alpha_w\beta_w\theta_w \ln x_{1,min}(t).\end{aligned} \tag{64}$$

Let $C + |R_w|\alpha_w\beta_w\theta_w\ln d_w + \beta_w B = H$, then we have

$$\dot{x}_{2,min}(t) \leq H - |R_w|\alpha_w\beta_w\theta_w \ln x_{1,min}(t). \tag{65}$$

Now the motion of the system is equivalent to a particle moving in a straight line, with the initial position, $x_{1,min}(t^0)$, initial speed, $x_{2,min}(t^0)$, and acceleration, $\dot{x}_{2,min}(t)$. Assume the mass of the particle is m. Then the force on the particle is equal to

$$F = \dot{x}_{2,min}(t)m. \tag{66}$$

The initial kinetic energy of the system

$$K = \frac{1}{2}m(x_{2,\,min}(t^0))^2. \tag{67}$$

When $x_{1,min}(t) \to 0^+$, the accumulated work done by the force approaches

$$\begin{aligned}W &= m\int_0^{x_{1,min}(t^0)} \dot{x}_{2,min}(x)dx \\ &\leq m\int_0^{x_{1,min}(t^0)} (H - |R_w|\alpha_w\beta_w\theta_w \ln x)dx \\ &= m(Hx_{1,min}(t^0) \\ &\quad - |R_w|\alpha_w\beta_w\theta_w x_{1,min}(t^0)\ln x_{1,min}(t^0) \\ &\quad + |R_w|\alpha_w\beta_w\theta_w x_{1,min}(t^0)).\end{aligned} \tag{68}$$

Suppose when $x_{1,min}(t) \to 0^+$, the system speed is v_{0^+}, from the energy conservation law

$$\begin{aligned}\frac{1}{2}m{v_{0^+}}^2 &= \frac{1}{2}m(x_{2,min}(t^0))^2 - W\\ &\geq \frac{1}{2}m(x_{2,min}(t^0))^2 - m(Hx_{1,min}(t^0)\\ &\quad - |R_w|\alpha_w\beta_w\theta_w x_{1,min}(t^0)\ln x_{1,min}(t^0)\\ &\quad + |R_w|\alpha_w\beta_w\theta_w x_{1,min}(t^0)).\end{aligned} \tag{69}$$

Thus, as long as the right-hand side of the inequality (69) is positive, that is

$$\begin{aligned}&x_{2,min}(t^0)\\ &> \sqrt{\begin{aligned}&2(Hx_{1,min}(t^0) - |R_w|\alpha_w\beta_w\theta_w x_{1,min}(t^0)\ln x_{1,min}(t^0)\\ &+ |R_w|\alpha_w\beta_w\theta_w x_{1,min}(t^0)).\end{aligned}}\end{aligned} \tag{70}$$

The path flow will penetrate 0 and turn negative.

We present a simple numerical example to demonstrate the existence of the negative flow when condition (70) is satisfied. The numerical case is conducted with the aid of the "NDSolve" function provided by *Mathematica* software. A very small "max step size" ($\Delta t = 10^{-4}$) is chosen to guarantee that the negative flow is not due to the discretization of the numerical solution. The example is conducted in a simple network that has a single OD pair served by two parallel links. The total demand is 50. The link performance functions are given by $c_1 = 15 + 1.5f_1$ for link 1 and $c_2 = 20 + 1.2f_2$ for link 2. The sensitivity coefficient α, the memory decay rate β, and the SUE dispersion parameter θ are fixed to be 3.5, 0.6, and 1, respectively.

To demonstrate the existence of negative path flow, we fix the initial link flow pattern $(f_1(t^0), f_2(t^0))$ to be $(25, 25)$ and vary the initial speed of link 1, $|\dot{f}_1(t^0)|$, from 0 to 90. Note that the equation $\dot{f}_1(t^0) + \dot{f}_2(t^0) = 0$ should be satisfied to guarantee the flow conservation. Figure 1 shows that the larger absolute initial speed would lead to a relatively more dramatic oscillation of link flows. When the initial speed is sufficiently large (In this numerical example, the critical value is $|\dot{f}_1(t^0)| = 60.31$), the path flow on link 1 penetrates 0. At this point the dynamical system (52)–(53) becomes invalid, because $\ln(\cdot)$ is not defined for negative values.

To guarantee that dynamical system (52)–(53) is always valid during the evolution process, we make an assumption below

Assumption 1.[1] *Define*

$$\Theta = \{\mathbf{x}_1 | \mathbf{x}_1 > \mathbf{0}, \Lambda\mathbf{x}_1 = \mathbf{d}\},$$

where $\mathbf{x}_1 > \mathbf{0}$ *means that all the elements of* $\mathbf{x}_1$ *are positive, and*

$$\Upsilon = \{\mathbf{x}_2 | x_{2,rw} \in \mathbb{R}, \textstyle\sum_{r\in R_w} x_{2,rw} = 0, \forall r \in R_w, w \in W\}.$$

We assume that, starting with any point in $\Theta \times \Upsilon$, *the trajectories of* $\mathbf{x}$ *under dynamical process (52)–(53) will always stay in* $\Theta \times \Upsilon$.

4.3. Stability of the Proposed Model

When the concept of path potential is applied to the second-order dynamical system considering the user-learning phenomenon, the fixed point transfers from DUE to SUE with the influence of the added term, and the stability analysis becomes more complicated than the first-order dynamic models. We first give the following proposition.

Lemma 7. $\mathbf{x}_1^* \in \Omega$ *coincides with SUE, if and only if* $\mathbf{x}^* = \begin{bmatrix}\mathbf{x}_1^*\\ \mathbf{0}\end{bmatrix}$ *is a fixed point of the dynamical system (52)–(53).*

Proof. The fixed point of dynamical system (52)–(53) can be expressed by

$$\begin{bmatrix}\dot{\mathbf{x}}_1^*\\ \dot{\mathbf{x}}_2^*\end{bmatrix} = \begin{bmatrix}\mathbf{x}_2^*\\ \alpha\beta\mathbf{g}(\mathbf{x}_1^*) - \beta\mathbf{x}_2^*\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}. \tag{71}$$

Then we have

$$\mathbf{x}_2^* = \mathbf{0}, \ \mathbf{g}(\mathbf{x}_1^*) = \mathbf{0}, \tag{72}$$

which yields

$$\sum_{s\in R_w}(\mu_{sw} - \mu_{rw}) = 0, \quad \forall r \in R_w. \tag{73}$$

Equations (73) holds if and only if

$$\mu_{sw} = \mu_{rw} \qquad \forall r, s \in R_w. \tag{74}$$

Using the definition of chemical potential (48) and rearranging the equation, we have

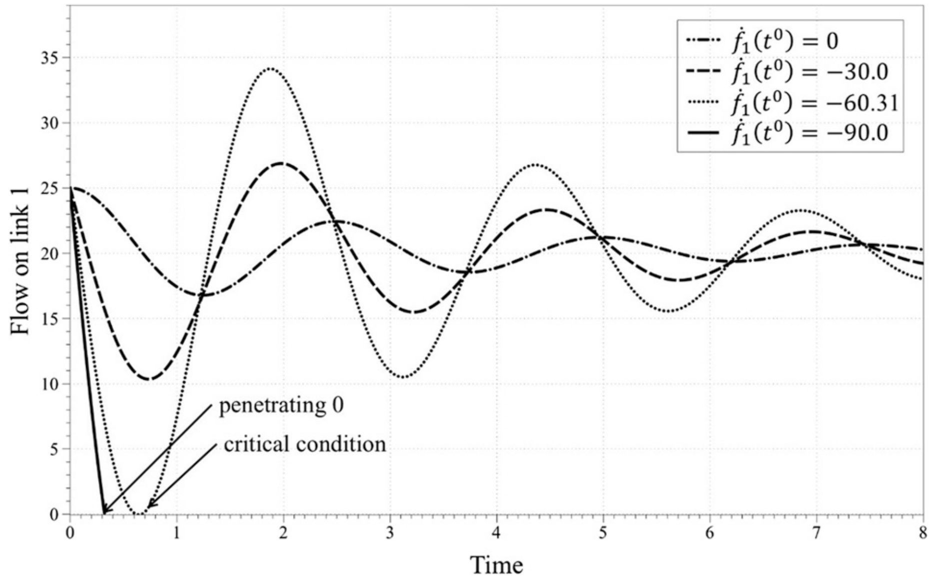
$$\frac{x_{1,rw}^*}{x_{1,sw}^*} = \frac{\exp(-c_{rw}/\theta_w)}{\exp(-c_{sw}/\theta_w)} \qquad \forall r, s \in R_w. \tag{75}$$

From Equation (75), $\mathbf{x}_1^*$ exactly coincides with SUE. □

Inspired by the results in Section 3, we establish the candidate Lyapunov function for the second-order SUE model

$$\begin{aligned}E(\mathbf{x}) &= G_{tra} + E_k\\ &= \sum_{a\in A}\int_0^{v_a(\mathbf{x}_1)} c_a(\omega)d\omega + \sum_{w\in W}\sum_{r\in R_w}\theta_w f_{rw}\ln(f_{rw})\\ &\quad + \sum_{w\in W}\sum_{r\in R_w}\frac{1}{2}m_w x_{2,rw}^2.\end{aligned} \tag{76}$$

Figure 1. Evolutions of Flow on Link-1 Under Different Initial Speeds, $\dot{f}_1(t^0)$



Lemma 8. *The minimum point of function $E(\mathbf{x})$ is obtained at $\mathbf{x}^* = \begin{bmatrix} \mathbf{x}_1^* \\ \mathbf{0} \end{bmatrix}$, where $\mathbf{x}_1^*$ is the SUE path flow.*

Proof. With the assumption that the cost function is strictly increasing, it is obvious that $E(\mathbf{x})$ is strictly convex, which implies that $E(\mathbf{x})$ has a unique minimum point. The Lagrangian is as follows

$$L(\mathbf{x}, \boldsymbol{\gamma}) = E(\mathbf{x}) + \sum_{w \in W} \gamma_w \Big(d_w - \sum_{r \in R_w} x_{1,rw}\Big). \tag{77}$$

The first-order optimality conditions are

$$\begin{aligned} x_{1,rw}^* \frac{\partial L(\mathbf{x}, \boldsymbol{\gamma})}{\partial x_{1,rw}^*} &= x_{1,rw}^* (c_{rw}^* + \theta_w \ln x_{1,rw}^* + \theta_w - \gamma_w{}^*) \\ &= 0, \ \forall r \in R_w, \ w \in W, \end{aligned} \tag{78}$$

$$c_{rw}^* + \theta_w \ln x_{1,rw}^* + \theta_w - \gamma_w{}^* = 0, x_{1,rw}^* > 0, \ \forall r \in R_w, \ w \in W, \tag{79}$$

$$\frac{\partial L(\mathbf{x}, \boldsymbol{\gamma})}{\partial x_{2,rw}^*} = m_w x_{2,rw}^* = 0 \qquad \forall r \in R_w, w \in W. \tag{80}$$

From Equations (79) and (80), we conclude that $\mathbf{x}_1^*$ is the SUE path flow and $\mathbf{x}_2^* = \mathbf{0}$. □

Proposition 2. *Under Assumption 1, the function*

$$V_2(x) = E(x) - \min E(x) \tag{81}$$

is a candidate Lyapunov function of dynamical system (52)–(53), and the dynamical system asymptotically converges to SUE.

Proof. Please refer to Online Appendix B.

5. Conclusion and Future Research

In this study, a general framework is established for the day-to-day dynamics converging to SUE, via the definition of path potential in a transportation network. Both the logit dynamic and logit-based Smith dynamic are contained in this framework, so that the connection between them is revealed. Moreover, we extend the traditional BNN dynamic and the second-order dynamic developed in Xiao et al. (2016) for SUE. By analogy to the concepts of Gibbs free energy in thermodynamics, we define the Gibbs free energy for a transportation network, which takes the form of Fisk's formulation. We prove that Fisk's formulation can be utilized as a general Lyapunov function for the first-order day-to-day dynamics for SUE, and the Lyapunov function for the second-order day-to-day dynamics for SUE is also found. A sufficient condition to prevent negative path flow for the first-order models is provided. However, for the second-order model, we find that the path flow could still go negative owing to the inertia of the system, even when the system converges to SUE with positive path flows.

This study provides another interesting example that the traditional concepts and principles in physics may have their counterparts in transportation area. Consequently, human's aggregate behavior may be governed by some physical laws. A bridge is built from the deterministic day-to-day models that converge to DUE to their stochastic counterparts by replacing the travel costs of the paths with their potentials. Through this bridge, numerous existing deterministic day-to-day models may potentially be extended to capture the randomness in travelers' route choice behavior in the future. The Lyapunov-stability of these models can be guaranteed by viewing Fisk's formulation as a candidate Lyapunov function, so far as they satisfy the condition of negative correlation, which requires the path flow growth rates to be negatively correlated with the path potentials.

Endnote

[1] This assumption is more relaxed than the assumption 1 in Xiao et al. (2016), because the path flows at SUE are always positive by definition.

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